

**UNIVERSITY  
OF OSLO**

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**Rigidity and amenability of locally compact  
groups via their actions on  $L^p$ -spaces**

**Thesis submitted for the degree of  
Philosophiae Doctor**

Department of Mathematics  
Faculty of Mathematics and Natural Sciences

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**2025**





*Når man føler hvor lidet  
man når med sit flid  
er det nyttigt at mindes, at  
Ting Tager Tid*

— Piet Hein



## Abstract (EN)

Analytic properties of locally compact groups, such as amenability and Kazhdan-type rigidity, offer deep insights into the structure and behavior of groups. It is well known that these properties are intimately connected with the isometric representation theory. The primary focus of this thesis is to study amenability and Kazhdan-type rigidity via representations by surjective isometries on  $L^p$ -spaces and via pseudofunction algebras generated from classes of such representations.

The thesis is a collection of four papers. Topics covered include  $L^p$ -integrability of representations induced from open subgroups, Kazhdan-type rigidity for Banach algebras in general and pseudofunction algebras in particular, equivalent definitions of property  $(T_{L^p})$  for discrete groups, and characterizations of amenability related to the symmetrized  $p$ -pseudofunction algebra.

## Resumé (DK)

Analytiske egenskaber af lokalkompakte grupper, såsom amenabilitet og rigiditet af Kazhdan-typen, tilbyder dyb indsigt ind i strukturen og opførslen af grupper. Det er velkendt at disse egenskaber er tæt forbundne med den isometriske representationsteori. Det primære fokus i denne afhandling er at studere amenabilitet og rigiditet af Kazhdan-typen via representationer ved surjektive isometrier på  $L^p$ -rum samt via pseudofunktionsalgebraer genereret fra klasser af sådanne representationer.

Afhandlingen er en samling af fire artikler. Emner der dækkes omfatter  $L^p$ -integrabilitet af representationer induceret fra åbne undergrupper, rigiditet af Kazhdan-typen for Banach algebraer og heraf specielt pseudofunktionsalgebraer, ækvivalente definitioner af egenskab  $(T_{L^p})$  for diskrete grupper, samt karakteriseringer af amenabilitet relateret til den symmetriserede  $p$ -pseudofunktionsalgebra.

## Résumé (FR)

Les propriétés analytiques des groupes localement compacts, telles que la moyennabilité et la rigidité de type Kazhdan, offrent des perspectives approfondies sur la structure et le comportement des groupes. Il est bien connu que ces propriétés sont intimement liées à la théorie des représentations isométriques. L'objectif principal de cette thèse est d'étudier la moyennabilité et la rigidité de type Kazhdan via des représentations par isométries surjectives sur des  $L^p$ -espaces et via des algèbres de pseudofonctions générées à partir de classes de telles représentations.

La thèse est un recueil de quatre articles. Les sujets abordés incluent l'intégrabilité  $L^p$  des représentations induites à partir de sous-groupes ouverts, la rigidité de type Kazhdan pour les algèbres de Banach en général et les algèbres de pseudo-fonctions en particulier, les définitions équivalentes de la propriété  $(T_{L^p})$  pour les groupes discrets, et les caractérisations de la moyennabilité liées à l'algèbre de  $p$ -pseudofonctions symétrisée.

## Sammendrag (NO)

Analytiske egenskaper til lokalt kompakte grupper, slik som amenabilitet og rigiditet av Kazhdan-typen, gir dyp innsikt i strukturen og oppførselen til grupper. Det er velkjent at disse egenskapene er nært knyttet til den isometriske representasjonsteorien. Hovedfokuset for denne avhandlingen er å studere amenabilitet og rigiditet av Kazhdan-typen via representasjoner av surjektive isometrier på  $L^p$ -rom, så vel som via pseudofunksjonalgebraer generert fra klasser av slike representasjoner.

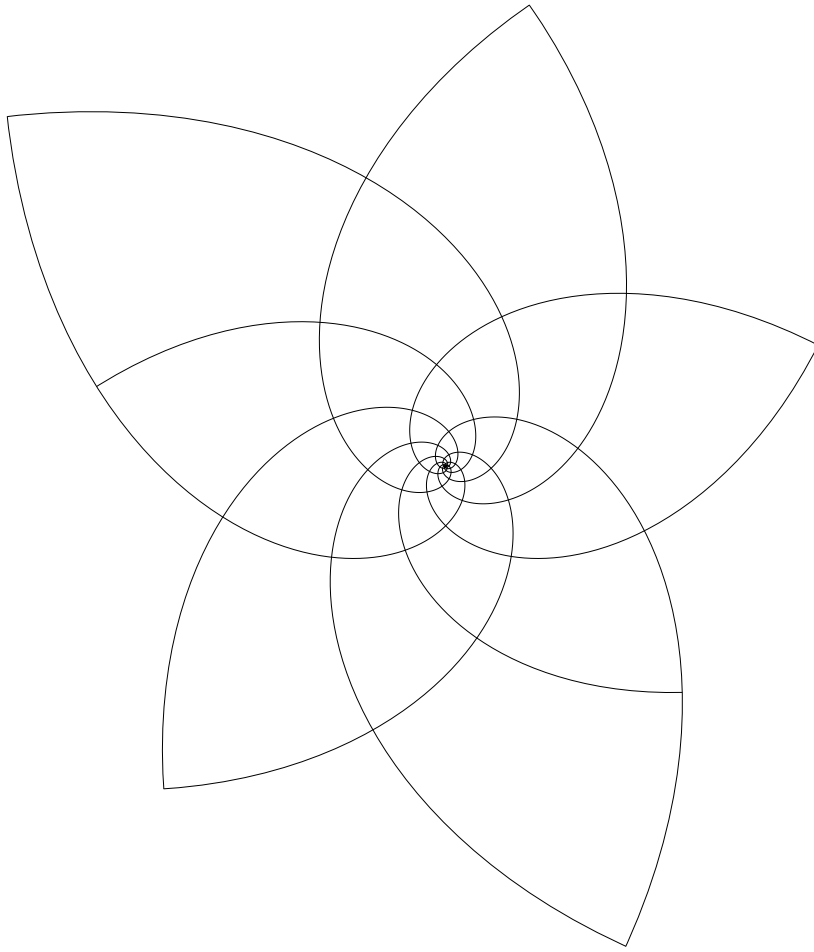
Avhandlingen er en samling av fire artikler. Emner som dekkes inkluderer  $L^p$ -integrerbarhet av representasjoner induisert fra åpne undergrupper, rigiditet av Kazhdan-typen for Banachalgebraer og derav spesielt pseudofunksjonalgebraer, ekvivalente definisjoner av egenskapen  $(T_{L^p})$  for diskrete grupper, og karakteriseringer av amenabilitet relatert til den symmetriserte  $p$ -pseudofunksjonalgebra.

## Zusammenfassung (DE)

Analytische Eigenschaften lokal kompakter Gruppen wie Amenabilität und Rigidität vom Kazhdan-Typ bieten tiefe Einblicke in die Struktur und das Verhalten von Gruppen. Es ist bekannt, dass diese Eigenschaften eng mit der isometrischen Darstellungstheorie verbunden sind. Der Hauptfokus dieser Arbeit liegt auf der Untersuchung von Amenabilität und Rigidität vom Kazhdan-Typ über Darstellungen durch surjektive Isometrien auf  $L^p$ -Räumen und über Pseudofunktionsalgebren, die aus Klassen solcher Darstellungen generiert werden.

Diese Dissertation ist eine Sammlung von vier Artikeln. Zu den behandelten Themen gehören  $L^p$ -Integrabilität von Darstellungen, die von offenen Untergruppen induziert werden, Rigidität vom Kazhdan-Typ für Banach-Algebren im Allgemeinen und Pseudofunktionsalgebren im Speziellen, äquivalente Definitionen der Eigenschaft  $(T_{L^p})$  für diskrete Gruppen und Charakterisierungen der Amenabilität im Zusammenhang mit der symmetrisierten  $p$ -Pseudofunktionsalgebra.







# Preface

This thesis is submitted in partial fulfillment of the requirements for the degree of *Philosophiae Doctor* at the University of Oslo. The research presented here was conducted at the University of Oslo under the supervision of Professor Nadia Larsen and CNRS senior researcher Mikael de la Salle.

The thesis is a collection of 4 papers, presented in chronological order of writing. This differs slightly from the order in which the research was conducted in that the fourth paper, although finished last, was initiated first. In particular, a significant part of the research presented in paper IV was conducted early on in the PhD process. The papers are preceded by a preliminary part, which serves to motivate our research and to put our results into a historical context, as well as to provide background necessary to read the papers.



# List of Papers

## Paper I

Antje Dabeler, Emilie Mai Elkiær, Maria Gerasimova & Tim de Laat: “Unitary  $L^{p+}$ -representations of almost automorphism groups”. In *Comptes Rendus Mathématique* 362 (2024), pp. 245–249. doi:10.5802/crmath.549. arXiv:2304.01079.

## Paper II

Emilie Mai Elkiær & Sanaz Pooya: “Property (T) for Banach algebras”. To appear in *Journal of Operator Theory*. arXiv:2310.18136.

## Paper III

Emilie Mai Elkiær: “Weak Property  $(T_{L^p})$  for discrete groups”. To appear in *Studia Mathematica*. doi:10.4064/sm240912-14-1. arXiv:2403.05312.

## Paper IV

Emilie Mai Elkiær: “Symmetrized pseudofunction algebras from  $L^p$ -representations and amenability of locally compact groups”. In *Expositiones Mathematicae* 43.4 (2025), p. 125685. doi:10.1016/j.exmath.2025.125685. arXiv:2411.07710.



# Acknowledgements

I want to thank my supervisor, Nadia Larsen. Thank you for the countless hours you have listened to my presentations with patience and for always demanding clarity. Thank you for carefully reading my writings and for comments that greatly improved its perspicuousness. Thank you for giving me the freedom to pursue my ideas and to find my own path.

Likewise, and in equal measure, I want to thank my co-supervisor, Mikael de la Salle, whose positive spirit and curiosity are inspirational. Thank you for the enlightening conversations and for sharing your mathematical insights with such generosity. Thank you for all the encouragement you gave me and for seeing the value in my work when I was doubtful of its significance and meaningfulness.

I am grateful to Makoto Yamashita, Eusebio Gardella, and Matthew Daws for agreeing to be on the committee for my thesis defense. Thank you for your time.

Without my co-authors Tim, Maria, Antje and Sanaz, my PhD experience would have been a lonely one. I am grateful to each one of you for being part of my journey. I am honored to have worked with you. Thank you for welcoming me in Münster, respectively, in Stockholm and Potsdam. Thank you all for all the conversations about math and beyond.

I am grateful for the Operator Algebras Mentor Network (OAMN), to those who made it come to be, and to those who keep it running. My deep-felt gratitude goes to all the mentors and fellow mentees I have met through this network. You know who you are. Thank you for letting me share with you my ups and downs and for sharing yours with me. Thank you for your support. A special thanks goes to Stefaan who encouraged me to apply for this PhD position in a moment where I did not feel strong enough to do so. I wish to also express my gratitude to Karen, Anna, Kari, Jennifer, Sarah, and Maria Paula. It was an honor to serve on the OAMN board with you, and I am grateful for all the thoughtful and meaningful conversations we had. You taught me a lot.

During my time as a PhD student, I have had opportunities to travel to several conferences and workshops. For this I am grateful. To everyone I have met during my travels and with whom I have shared many joyful moments: It has been a pleasure to meet and get to know you.

Among my peers at the University of Oslo, a special thank you goes to my office mate Francesca for getting me to start dancing again. Further, a thank you goes to Floris for co-creating the ‘What is..?’-seminar series with me. Thank you, Victor and Adé for helping me run it. Thank you, Anne, Elias, and Jonas for keeping it going. And, of course, thank you to everyone who shared fun pieces of

mathematics and everyone who joined and hereby made the ‘What is..?’ into a community.

I want to express my deep gratitude to my parents, whose support has been unconditional. My path from the day I first stepped into university until now where I am about to hand in this document has been long and winding. You may have not always understood the decisions I made. But through all these years, through highs and lows and across country borders, you have stood by me when I needed it. From the depths of my heart, thank you.

To my husband, Alex. In the times when I have most doubted my abilities, your mantra is what kept my confidence up: *When you think about something for long enough, something will come out.* Thank you for always believing that this mantra also applies to me, for believing that I am capable of doing research and for the many times you have let me know this. Thank you for being you together with me.

Til min datter, Linn, min stjerne. Tak for din kærlighed.

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# **Introduction & preliminaries**



# Introduction

Given a locally compact group  $G$  with left Haar measure  $\mu_G$ , consider the space  $C_c(G)$  consisting of complex valued continuous functions on  $G$  with compact support. In a natural way, this is an algebra over  $\mathbb{C}$  with addition and scalar multiplication defined pointwise and with multiplication given by convolution. A basic construction starting from the algebra  $C_c(G)$  is to equip it with an appropriate Banach algebra norm and complete it. Notable examples of this construction are the universal and the reduced group  $C^*$ -algebras, denoted by  $C^*(G)$  and  $C_r^*(G)$ , respectively. Other well known examples include the  $p$ -pseudofunction algebras  $F_{\lambda_p}(G)$  introduced by Herz in [Her71] which, like the reduced group  $C^*$ -algebra, are generated by the left-regular representation but on  $L^p(G)$ . More generally, any class of isometric representations of  $G$  give rise to a semi-norm on  $C_c(G)$ , and the completion of the appropriate quotient of  $C_c(G)$  will then be a Banach algebra. We shall refer to these Banach algebras under one umbrella as *pseudofunction algebras*. Now, what can be learned about the group from the pseudofunction algebras it generates? The idea that properties which can be understood in representation theoretic terms could be recognized by these objects is not far fetched. After all, each pseudofunction algebra is constructed using a specific part of the isometric representation theory of  $G$ , e.g.,  $C^*(G)$  is constructed from the unitary representation theory and  $C_r^*(G)$  and  $F_{\lambda_p}(G)$  from the left-regular representation. It is well known that analytic properties such as Kazhdan's property (T) and amenability, which are intimately connected to the unitary representation theory of the group, can be understood in terms of the group  $C^*$ -algebras. The underlying motive in this thesis is to study aspects of the interplay between analytic properties of the group in question and properties of the pseudofunction algebras it generates. We shall focus mainly on Kazhdan-type rigidity properties and on amenability, and we are, in particular, interested in pseudofunction algebras coming from isometric representations on  $L^p$ -spaces when  $p \neq 2$ .

Property (T) originates with Kazhdan in his 3 page long paper [Kaz67] from 1967. It describes a rigidity in how a group may act unitarily on a complex Hilbert space. In Kazhdan's original definition, a locally compact group has property (T) if the trivial representation of  $G$  is isolated in the Fell topology on the unitary dual of  $G$ . In more basic representation theoretic terms, it is the property that, for every unitary representation of  $G$ , nets of almost  $G$ -invariant unit vectors must approach the subspace of  $G$ -invariant vectors in an appropriate sense (see section 3.2 for a precise definition). Examples of groups with property (T) include compact groups and  $SL_n(\mathbb{K})$  and  $Sp_{2n}(\mathbb{K})$  with  $n \geq 3$  and  $\mathbb{K}$  a local field, i.e., a non-discrete locally

compact field. Further, any lattice in a locally compact group  $G$  has property (T) if and only if  $G$  does. Kazhdan's motivation to introduce property (T) was as a tool to show that certain lattices are finitely generated. Since its introduction, it has proved itself more than just a tool with a single use. It is now a fundamental concept in analytic group theory with connection to several areas of mathematics. For example, it was used by Margulis in his construction of expander graphs. And in ergodic group theory, deep results by Schmidt in [Sch80] and Connes and Weiss in [CW80] show that property (T) is equivalent to the property that every ergodic p.m.p. action is necessarily strongly ergodic. This is just to mention a few of its many connections. We refer the reader to [BLV08] for a comprehensive introduction to the topic including historical notes.

In this thesis, we are interested in generalizations of property (T). It is one of these games we mathematicians like to play: If something is interesting in one category, maybe it is also interesting in another. For example, one may ask what would be a reasonable definition of property (T) if we interchange Hilbert spaces with Banach spaces? Or groups with von Neumann algebras,  $C^*$ -algebras or other algebras? We refer to such generalizations as Kazhdan-type rigidity properties.

The first paper in which property (T) was studied in connection with group actions on general Banach spaces appeared in 2003. This was the paper [FM03] by Fisher and Margulis who studied actions of property (T) groups on  $L^p$ -spaces. A Banach space analogue of property (T) was introduced shortly hereafter by Bader, Furman, Gellander and Monod in their seminal paper [BFGM07] from 2007. Their property  $(T_{\mathcal{E}})$  is defined with respect to a class of Banach spaces  $\mathcal{E}$  and can be thought of as a rigidity property concerning how a locally compact group may act by isometries on spaces from the the class  $\mathcal{E}$ .

Property  $(T_{\mathcal{E}})$  of Bader, Furman, Gellander and Monod is modeled on a representation theoretic definition of Kazhdan's property (T). But here there are a few different equivalent definitions to choose from. One may ask if a different choice still yields the same property in the Banach space setting. We shall clarify a few equivalences in section 3.2, and in paper III, we address one question in this direction concerning an a priori weaker version of property  $(T_{\mathcal{E}})$  when  $\mathcal{E}$  is the class of  $L^p$ -spaces.

In the realm of operator algebras, property (T) was introduced for von Neumann algebras in the 1980's; first for  $II_1$ -factors by Connes in [Con80] and shortly hereafter for general von Neumann algebras by Connes and Jones in [CJ85]. Property (T) for  $C^*$ -algebras did not appear in the literature before the early 2000's. In [Bek05], Bekka defines a version of property (T) for unital  $C^*$ -algebras, and in [Ng13] by Ng, two versions of property (T) are defined for general  $C^*$ -algebras: a weaker and a stronger version. Subsequently, in joint work of Bekka and Ng [BN19], it is shown that these definitions are sensible in that both versions of property (T) for the universal group  $C^*$ -algebra are equivalent to Kazhdan's property (T) for the group.

Paper II in this thesis can be seen as a unified approach to the two generalizations of property (T) discussed above. Here, we define two versions of property  $(T_{\mathcal{E}})$  for a large class of Banach algebras. When the Banach algebra in question is a  $C^*$ -algebra and  $\mathcal{E}$  is the class of complex Hilbert spaces, we recover the two

versions of property (T) for  $C^*$ -algebras in the sense of Bekka and Ng. Property  $(T_{\mathcal{E}})$  in the sense of Bader, Furman, Gelander and Monod for a locally compact group  $G$  is recovered as property  $(T_{\mathcal{E}})$  for the  $\mathcal{E}$ -pseudofunction algebra on  $G$ .

Another interesting analytic property for locally compact groups much older than Kazhdan's property (T) is the notion of amenability. Amenability was introduced by von Neumann in [Neu29] in the context of measure theory. After Lebesgue's work on integration theory in the early 1900's [Leb04], Hausdorff raised the question if the Lebesgue measure on  $\mathbb{R}^n$  could be extended to a finitely additive measure on all subsets of  $\mathbb{R}^n$  and answered the question negatively for  $n \geq 3$  [Hau14]. It was subsequently shown by Banach that the question has an affirmative answer when  $n$  equals 1 or 2 [Ban23]. Von Neumann's insight was that the difference between the situation when  $n$  is small and when  $n$  is 3 or larger lie not in the structure of  $\mathbb{R}^n$  but in the structure of its isometry group. In particular, the deep reason for this difference is that the isometry group of  $\mathbb{R}^n$  is amenable for  $n = 1$  and 2 while for  $n \geq 3$  it is not. A locally compact group  $G$  is said to be amenable in the sense of von Neumann if there exists a left-invariant mean on  $L^\infty(G)$ . Von Neumann himself named this property *meßbar*. The term amenability was coined by Day in his abstract for a presentation given at the summer meeting of the American Mathematical Society held in Boulder in 1949 [Day49]. Word has it that it was meant as a pun: Groups admitting a left-invariant *mean* are truly *amenable* in the colloquial sense of the word.

Just like property (T), amenability is an important analytic property with connections to several different areas of mathematics. Most important for us is the connection to representation theory. Here, a deep result of Hulanicki and Reiter establishes the following characterization: A locally compact group is amenable if and only if its trivial representation is weakly contained in its left regular representation in the sense of Fell. In  $C^*$ -algebraic terms, this translates to the property that the trivial representation of the group in question extends to a  $*$ -representation of its reduced group  $C^*$ -algebra, or, equivalently, that its reduced and universal group  $C^*$ -algebras coincide canonically. Further, the dual of the  $C^*$ -algebraic picture touches upon abstract harmonic analysis: A locally compact group is amenable if and only if the Banach space dual of its reduced group  $C^*$ -algebra agrees with its Fourier-Stieltjes algebra. In paper IV, we explore  $L^p$ -generalizations of the characterizations highlighted here.

When a locally compact group  $G$  is not amenable, so that the canonical contractive  $*$ -homomorphism from  $C^*(G)$  to  $C_r^*(G)$  is not injective, the question if there is something in between these two  $C^*$ -completions of  $L^1(G)$  begs to be asked. More precisely, given a  $C^*$ -norm  $\|\cdot\|_\mu$  on  $L^1(G)$  which is bounded from below and from above by the reduced, respectively, universal  $C^*$ -norm, the identity map on  $L^1(G)$  always extends to canonical surjective  $*$ -homomorphisms

$$C^*(G) \longrightarrow \overline{L^1(G)}^{\|\cdot\|_\mu} \longrightarrow C_r^*(G).$$

The  $C^*$ -norm  $\|\cdot\|_\mu$  is said to be *exotic* if both of these maps are not injective. In that case, we say that the completion of  $L^1(G)$  with respect to  $\|\cdot\|_\mu$  is an exotic

$C^*$ -completion of  $G$ . Potentially exotic  $C^*$ -norms may be constructed by taking the supremum over  $C^*$ -norms coming from some specified subclass of the unitary representations which contains the regular representation. A particularly nice way of doing this is given by Brown and Guentner in [BG13]. Here, a unitary representation is called an  $L^p$ -representation if sufficiently many of its matrix coefficients are  $L^p$ -integrable. Similarly, it is an  $L^{p^+}$ -representation if it is an  $L^{p+\varepsilon}$ -representation, for all  $\varepsilon > 0$ . The  $L^p$ -, respectively,  $L^{p^+}$ - $C^*$ -algebra of  $G$  is then the completion of  $L^1(G)$  with respect to the norm arising as the supremum over all unitary  $L^p$ -, respectively,  $L^{p^+}$ -representations. In paper I, we study integrability properties of representations induced from  $L^p$ - and  $L^{p^+}$ -representations of open subgroups of locally compact groups.

In the preliminary part of this thesis, we introduce and elaborate on technical details of constructions which are of central importance for the research presented in the second and main part of the thesis. In Chapter 1, we define and discuss two constructions associated to classes of isometric representations of locally compact groups: pseudofunction algebras and spaces of matrix coefficients. In Chapter 2, we give a thorough introduction to the famous Banach-Lamperti theorem, which characterizes the surjective isometries of  $L^p$ -spaces when  $p \neq 2$ . In Chapter 3, we discuss amenability and property (T). Finally, in Chapter 4, we provide summaries of the four papers constituting the thesis.

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# Chapter 1

## Algebras from isometric representations of locally compact groups

At its core, our aim is to study a locally compact group  $G$  through its actions on Banach spaces via isometries. One way to do this, which is central to the approach of this thesis, is to construct various Banach algebras on  $G$  associated to its isometric representation theory. We begin this chapter in Section 1.1 by briefly recalling the basics of isometric representations. In Section 1.2, we construct a family of Banach algebras which we refer to as pseudofunction algebras. These algebras will play a central role in paper II and paper IV. In Section 1.3, we discuss algebras of matrix coefficients that will appear in paper IV.

### 1.1 Isometric representations

Let  $G$  be a locally compact group. In this section we define what an isometric representation of  $G$  on a Banach space is and we recall the connection between the isometric representation theory of  $G$  and the non-degenerate contractive representation theory of  $L^1(G)$ . The material presented here can be considered folklore.

For a Banach space  $E$ , we denote by  $\mathcal{L}(E)$  the space of bounded linear operators on  $E$  and by  $\text{Isom}(E)$  the group of surjective linear isometries on  $E$ .

**Definition 1.1.1.** An *isometric representation* of a locally compact group  $G$  is a tuple  $(\pi, E)$  consisting of a Banach space  $E$  and a strongly continuous group homomorphism  $\pi : G \rightarrow \text{Isom}(E)$ .

*Remark 1.1.2.* Although we define an isometric representation to be a tuple  $(\pi, E)$ , we shall often use the term “isometric representation” to refer directly to the group homomorphism  $\pi$ .

*Remark 1.1.3.* When  $E$  is a complex or real Hilbert space, an isometric representation is commonly known as a unitary, respectively, orthogonal representation.

**Example 1.1.4.** Important isometric representations of a locally compact group  $G$  include the following:

- (i) The trivial representation  $(1_G, \mathbb{C})$  is given by  $1_G(t) = 1$ , for  $t \in G$ .
- (ii) For  $p$  a real number with  $1 \leq p < \infty$ , the left-regular representation  $(\lambda_p, L^p(G))$  is given by, for  $s, t \in G$  and  $\xi \in L^p(G)$ ,

$$\lambda_p(t)\xi(s) = \xi(t^{-1}s)$$

When  $p = 2$ , it is common to omit the subscript and write  $\lambda = \lambda_2$ . ◦

Given two isometric representations  $(\pi, E)$  and  $(\rho, F)$ , a linear map  $T : E \rightarrow F$  is called an *intertwiner*, or is said to be *equivariant*, if  $T\pi(t)\xi = \rho(t)T\xi$ , for all  $t \in G$  and  $\xi \in E$ . Equivariant isometries are the natural choice of morphisms in the category of isometric representations. We say that  $\pi$  is *contained in*  $\rho$ , and write  $\pi \leq \rho$ , if there exists an equivariant isometry  $E \rightarrow F$ . Further, we say that  $\pi$  and  $\rho$  are *equivalent* if each is contained in the other, or, equivalently, if there exists an equivariant surjective isometry between  $E$  and  $F$ .

**Definition 1.1.5.** A *contractive representation*  $(\pi, E)$  of the Banach algebra  $\mathcal{A}$  on the Banach space  $E$  is a contractive Banach algebra homomorphism  $\mathcal{A} \rightarrow \mathcal{L}(E)$ . We say that  $\pi$  is *non-degenerate* if the linear span of  $\mathcal{A}E$  is dense in  $E$ .

For a locally compact group  $G$  with left-invariant Haar measure, the linear space  $L^1(G)$  is a Banach algebra under the convolution product given by, for  $f, g \in L^1(G)$  and  $t \in G$ ,

$$(f * g)(t) = \int_G f(s)g(s^{-1}t) d\mu_G(s).$$

The non-degenerate contractive representations of  $L^1(G)$  are intimately connected with the isometric representations of  $G$ . Indeed, any isometric representation  $\pi$  of  $G$  on a Banach space  $E$  integrates to a non-degenerate contractive representation of  $L^1(G)$  on  $E$ : For each  $f \in L^1(G)$  and each  $\xi \in E$ , set

$$\pi(f)\xi = \int_G f(s)\pi(s)\xi d\mu_G(s).$$

We refer to this representation as the *integrated form* of  $\pi$ . We adopt the common abuse of notation and use  $\pi$  to denote both an isometric representation and its integrated form. We record in Proposition 1.1.6 the folklore fact that any non-degenerate contractive representation of  $L^1(G)$  arise as the integrated form of an isometric representation.

**Proposition 1.1.6.** *Let  $G$  be a locally compact group and  $E$  a Banach space. There is a 1-1 correspondence between non-degenerate contractive representations of  $L^1(G)$  on  $E$  and isometric representations of  $G$  on  $E$ .*

## 1.2 Pseudofunction algebras

By the term *pseudofunction algebra* on the locally compact group  $G$ , we mean a Banach algebra completion of the convolution algebra  $L^1(G)$  with a norm coming from a class of isometric representations of  $G$ . In this section, we will make this construction precise and discuss some basic properties of these algebras. The  $p$ -pseudofunction algebra, which will appear as a special case of the construction presented here, was introduced by Herz in the 1970's in [Her71]. The construction of general pseudofunction algebras can be found in, e.g., [Run04], [GT14] and [DN15].

Let  $\mathcal{R}$  be a non-empty class of isometric representations of  $G$ . For each  $f \in L^1(G)$ , set

$$\|f\|_{\mathcal{R}} = \sup \{ \|\pi(f)\| \mid \pi \in \mathcal{R} \}.$$

This supremum exists also when  $\mathcal{R}$  is not a set because it is the supremum of a subset of  $\mathbb{R}$ . Further it is bounded above by the 1-norm of  $f$ . In general,  $\|\cdot\|_{\mathcal{R}}$  always defines a submultiplicative seminorm. If  $\mathcal{R}$  *separates the points of*  $L^1(G)$ , i.e., if for every non-zero  $f \in L^1(G)$  there exists  $\pi \in \mathcal{R}$  such that  $\pi(f) \neq 0$ , then  $\|\cdot\|_{\mathcal{R}}$  defines a submultiplicative norm on  $L^1(G)$ . Set

$$I_{\mathcal{R}} = \bigcap_{\pi \in \mathcal{R}} \ker(\pi) = \left\{ f \in L^1(G) \mid \|f\|_{\mathcal{R}} = 0 \right\}.$$

Observe that  $I_{\mathcal{R}}$  is a closed 2-sided ideal in  $L^1(G)$ , and so, the quotient  $L^1(G)/I_{\mathcal{R}}$  inherits the algebra structure from  $L^1(G)$ . We denote by  $F_{\mathcal{R}}(G)$  the completion of  $L^1(G)/I_{\mathcal{R}}$  with respect to the norm induced from  $\|\cdot\|_{\mathcal{R}}$  and refer to it as the *Banach algebra of  $\mathcal{R}$ -pseudofunctions* or, simply, the  *$\mathcal{R}$ -pseudofunction algebra*. It is a Banach algebra with multiplication extending the convolution product on the quotient  $L^1(G)/I_{\mathcal{R}}$ . Whenever using  $\mathcal{R}$  as the subscript for the norm on  $F_{\mathcal{R}}(G)$  could cause confusion, we shall denote the norm instead by  $\|\cdot\|_{F_{\mathcal{R}}}$ .

We are interested in two special classes of  $\mathcal{R}$ -pseudofunction algebras: When  $\mathcal{R}$  is either the class  $\text{Rep}_{\mathcal{E}}(G)$  of all isometric representations on a given class of Banach spaces  $\mathcal{E}$ , or when  $\mathcal{R}$  consists of a single representation  $\pi$ . In these cases, we simplify notation and denote by  $F_{\mathcal{E}}(G)$ , respectively,  $F_{\pi}(G)$  the resulting Banach algebra. Accordingly, we refer to these Banach algebras as algebras of  *$\mathcal{E}$ -pseudofunctions*, respectively,  *$\pi$ -pseudofunctions*.

**Example 1.2.1.** Well-known examples of pseudofunction algebras include:

- (i) The universal group  $C^*$ -algebra  $C^*(G)$  is the pseudofunction algebra  $F_{\mathcal{H}}(G)$ , where  $\mathcal{H}$  is the class of complex Hilbert spaces.
- (ii) The reduced group  $C^*$ -algebra  $C_r^*(G)$  is the pseudofunction algebra  $F_{\lambda}(G)$ , where  $\lambda$  denotes the left-regular representation of  $G$  on  $L^2(G)$ .
- (iii) For  $1 \leq p < \infty$ , the  $p$ -pseudofunction algebra of Herz [Her71], often denoted by  $PF_p(G)$  in the literature, is the pseudofunction algebra  $F_{\lambda_p}(G)$ , where  $\lambda_p$  denotes the left-regular representation of  $G$  on  $L^p(G)$ .

- (iv)  $L^1(G)$  is itself a pseudofunction algebra. Indeed, the 1-pseudofunction algebra of item (iii) recovers  $L^1(G)$ .  $\circ$

Fix a class  $\mathcal{R}$  of isometric representations of  $G$ . We gather in the following a few facts about the  $\mathcal{R}$ -pseudofunction algebra of  $G$ . Observe that, for any  $\pi \in \mathcal{R}$  and any  $f \in L^1(G)$ , we have

$$\|\pi(f)\| \leq \|f\|_{\mathcal{R}} \leq \|f\|_1. \quad (1.1)$$

With these inequalities, it is straightforward to verify Proposition 1.2.2 relating  $F_{\mathcal{R}}(G)$  to  $L^1(G)$  and Proposition 1.2.3 relating it to pseudofunction algebras coming from subclasses of  $\mathcal{R}$ .

**Proposition 1.2.2.** *The canonical map  $L^1(G) \rightarrow F_{\mathcal{R}}(G)$  which sends a function  $f \in L^1(G)$  to its equivalence class in  $F_{\mathcal{R}}(G)$  is a contractive homomorphism with dense range. It is injective if and only if  $\mathcal{R}$  separates the points of  $L^1(G)$ .*

**Proposition 1.2.3.** *For any subclass  $\mathcal{S} \subset \mathcal{R}$ , the identity on  $L^1(G)$  extends to a contractive homomorphism  $F_{\mathcal{R}}(G) \rightarrow F_{\mathcal{S}}(G)$  with dense range.*

Proposition 1.2.4, which follows from the first inequality of equation (1.1), establishes that the  $\mathcal{R}$ -pseudofunction algebra is universal for the class  $\mathcal{R}$  in the same way that  $C^*(G)$  is universal for the unitary representations of  $G$ .

**Proposition 1.2.4** ( $\mathcal{R}$ -universality). *Every representation  $\pi \in \mathcal{R}$  extends to a non-degenerate contractive representation of  $F_{\mathcal{R}}(G)$ .*

Given a non-degenerate contractive representation  $\pi$  of  $F_{\mathcal{R}}(G)$ , observe that  $\pi$  necessarily extends the integrated form of an isometric representation of  $G$ , by Proposition 1.1.6. We are, however, not guaranteed that  $\pi$  lies in the class  $\mathcal{R}$ , or even that it is contained in a representation from  $\mathcal{R}$ . That is, we do not have a converse to Proposition 1.2.4. We say that  $\pi$  is *weakly contained in  $\mathcal{R}$* , and write  $\pi \prec \mathcal{R}$ , when  $\|\pi(f)\| \leq \|f\|_{\mathcal{R}}$ , for all  $f \in L^1(G)$ .

**Proposition 1.2.5.** *Let  $(\pi, E)$  be an isometric Banach space representation of the locally compact group  $G$ . Then  $\pi$  extends to a non-degenerate contractive representation of  $F_{\mathcal{R}}(G)$  if and only if  $\pi$  is weakly contained in  $\mathcal{R}$ .*

*Proof.* The ‘only if’-part is direct from the definition of weak containment. For the ‘if’-part, assume that  $\pi \prec \mathcal{R}$ . Then  $\pi$  factors as a contractive algebra homomorphism defined on the image of  $L^1(G)$  inside  $F_{\mathcal{R}}(G)$ , and the resulting map extends to a contraction defined on all of  $F_{\mathcal{R}}(G)$ .  $\square$

The Banach algebra  $L^1(G)$  carries a natural isometric involution given by, for  $f \in L^1(G)$  and  $t \in G$ ,

$$f^*(t) = \Delta(t^{-1})\overline{f(t^{-1})}.$$

This makes  $L^1(G)$  a Banach  $*$ -algebra. But the involution on  $L^1(G)$  need not in general extend to  $F_{\mathcal{R}}(G)$ . However, when the class  $\mathcal{R}$  is closed under duality and conjugation, the involution does extend. We establish this in Proposition 1.2.8. This fact appears without proof in [DN15] and it is proven in Proposition 4.2 of

[SW20] in the special case when  $\mathcal{R}$  consists of the left regular representations on  $L^p(G)$  and on  $L^{p'}(G)$ , with  $1 < p < p' < \infty$  being Hölder conjugate. The proof we give is essentially the same as the one given in [SW20]. First, we make precise what we mean when we say that the class is *closed under duality*, respectively, *closed under conjugation*. We begin with the former.

For a Banach space  $E$ , we denote by  $E'$  its dual space. Given an isometric representation  $(\pi, E)$  of  $G$ , its *dual representation* is the isometric representation  $\pi'$  on  $E'$  defined by, for  $t \in G$ ,  $\eta \in E'$  and  $\xi \in E$ ,

$$\pi'(t)\eta(\xi) = \eta(\pi(t^{-1})\xi).$$

We say that the class  $\mathcal{R}$  is *closed under duality* if, for each  $\pi$  in the class  $\mathcal{R}$ , the dual representation  $\pi'$  is equivalent to a representation in  $\mathcal{R}$ . For an operator  $T \in \mathcal{L}(E)$ , denote by  $T^*$  the adjoint operator on  $E'$ . If  $T$  is an isometry, then so is  $T^*$ . We now have two ways of constructing isometries on  $E'$  via the representation  $(\pi, E)$ : We can use the dual representation or we can take the Banach space adjoints of the operators from the original representation. These two methods should not be confused, but they are connected as is shown in Lemma 1.2.6. For  $f \in L^1(G)$ , denote by  $\tilde{f} \in L^1(G)$  the function given by  $\tilde{f}(t) = \Delta(t^{-1})f(t^{-1})$ , for  $t \in G$ .

**Lemma 1.2.6.** *Let  $G$  be locally compact group, let  $(\pi, E)$  be an isometric representation and let  $(\pi', E')$  be its dual representation. Then  $\pi(f)^* = \pi'(\tilde{f})$ , for each  $f \in L^1(G)$ .*

*Proof.* For  $f \in L^1(G)$ ,  $\xi \in E$  and  $\eta \in E^*$ ,

$$\begin{aligned} (\pi(f)^*\eta)(\xi) &= \eta(\pi(f)\xi) = \int_G f(s)\eta(\pi(s)\xi) \, d\mu_G(s) \\ &= \int_G f(s)(\pi'(s^{-1})\eta)(\xi) \, d\mu_G(s) \\ &= \int_G \Delta(s^{-1})f(s^{-1})(\pi'(s)\eta)(\xi) \, d\mu_G(s) \\ &= \int_G \tilde{f}(s)(\pi'(s)\eta)(\xi) \, d\mu_G(s) \\ &= (\pi'(\tilde{f})\eta)(\xi). \quad \square \end{aligned}$$

For a linear space  $E$  over  $\mathbb{C}$ , its conjugate space  $\overline{E}$  is the space which is identical to  $E$  as an additive group, but with scalar multiplication defined by  $\alpha \cdot \xi = \overline{\alpha}\xi$ , for  $\alpha \in \mathbb{C}$  and  $\xi \in E$ . Given an isometric representation  $(\pi, E)$  of  $G$ , its *conjugate representation* is the isometric representation  $\overline{\pi}$  on  $\overline{E}$  which acts as  $\pi$  as a set theoretic transformation. The class  $\mathcal{R}$  is *closed under conjugation* if, for each  $\pi$  in the class  $\mathcal{R}$ , the conjugate representation  $\overline{\pi}$  is equivalent to a representation in  $\mathcal{R}$ . For  $f \in L^1(G)$ , denote by  $\overline{f}$  the function given by  $\overline{f}(t) = \overline{f(t)}$ , for  $t \in G$ .

**Lemma 1.2.7.** *Let  $G$  be locally compact group, let  $(\pi, E)$  be an isometric representation and let  $(\overline{\pi}, \overline{E})$  be its conjugate representation. Then  $\overline{\pi}(f) = \pi(\overline{f})$ , for each  $f \in L^1(G)$ .*

*Proof.* For  $f \in L^1(G)$  and  $\xi \in E$ ,

$$\overline{\pi}(f)\xi = \int_G f(s) \cdot (\overline{\pi}(s)\xi) \, d\mu_G(s) = \int_G \overline{f(s)}\pi(s)\xi \, d\mu_G(s) = \pi(\overline{f})\xi. \quad \square$$

**Proposition 1.2.8.** *Let  $\mathcal{R}$  be a class of isometric representations of  $G$  which is closed under duality and conjugation. The involution on  $L^1(G)$  is an isometry with respect to the seminorm associated with  $\mathcal{R}$ . In particular,  $F_{\mathcal{R}}(G)$  is a Banach \*-algebra.*

*Proof.* By assumption, if  $\pi$  lies in  $\mathcal{R}$  so does  $\overline{\pi'}$ . Using Lemma 1.2.6, Lemma 1.2.7 and the assumption that  $\mathcal{R}$  is closed under duality and conjugation, we obtain, for each  $f \in L^1(G)$ ,

$$\begin{aligned} \|f\|_{\mathcal{R}} &= \sup \{ \|\pi(f)\| \mid \pi \in \mathcal{R} \} \\ &= \sup \{ \|\pi(f)^*\| \mid \pi \in \mathcal{R} \} \\ &= \sup \{ \|\pi'(\tilde{f})\| \mid \pi \in \mathcal{R} \} \\ &= \sup \{ \|\overline{\pi'}(f^*)\| \mid \pi \in \mathcal{R} \} \\ &= \sup \{ \|\pi(f^*)\| \mid \pi \in \mathcal{R} \} \\ &= \|f^*\|_{\mathcal{R}}. \quad \square \end{aligned}$$

### 1.2.1 Symmetrization of pseudofunction algebras

Let  $G$  be a locally compact group, let  $\mathcal{R}$  a class of isometric representations of  $G$  and assume that  $\mathcal{R}$  is contained in a class of isometric representations of  $G$  containing  $\mathcal{R}$  and which is closed under duality and conjugation. We denote by  $\mathcal{R}^*$  the smallest such class and refer to it as a symmetrization of  $\mathcal{R}$ . For a non-discrete locally compact group, we are not ensured that any given class of isometric representations allow such a symmetrization. Indeed, when  $G$  is not discrete, the dual of an isometric representation may not be strongly continuous. Under the assumption that a symmetrization of  $\mathcal{R}$  is possible, we introduce a symmetrized version of the  $\mathcal{R}$ -pseudofunction algebra as follows: The *symmetrized  $\mathcal{R}$ -pseudofunction algebra* of  $G$ , denoted by  $F_{\mathcal{R}^*}^*(G)$ , is the pseudofunction algebra associated with  $\mathcal{R}^*$ , that is,  $F_{\mathcal{R}^*}^*(G) = F_{\mathcal{R}^*}(G)$ . Proposition 1.2.8 ensures that  $F_{\mathcal{R}^*}^*(G)$  is always a Banach \*-algebra.

**Example 1.2.9.** Let  $\mathcal{R}$  be a class of isometric representations on reflexive Banach spaces and assume that  $\mathcal{R}$  is closed under conjugation. Let  $\mathcal{R}'$  be the class of Banach spaces which are dual to a space in  $\mathcal{R}$ . If  $\mathcal{R}'$  is also closed under conjugation then  $\mathcal{R}^* = \mathcal{R} \cup \mathcal{R}'$  and, for  $f \in L^1(G)$ ,

$$\|f\|_{F_{\mathcal{R}^*}^*} = \max\{\|f\|_{\mathcal{R}}, \|f\|_{\mathcal{R}'}\}. \quad \circ$$

In the remaining of this section, we discuss symmetrization of pseudofunction algebras from representations on  $L^p$ - and  $QSL^p$ -spaces. We treat the  $L^p$ -pseudofunction algebra in Example 1.2.11, the  $p$ -pseudofunction algebra in Example 1.2.12, and the  $QSL^p$ -pseudofunction algebra in Example 1.2.17. In the following, let  $G$  be a locally compact group, fix  $1 < p < \infty$ , and let  $p'$  be the Hölder conjugate of  $p$ .

Consider an  $L^p$ -space  $L^p(\Omega, \nu)$ . For a function  $\xi \in L^p(\Omega, \nu)$ , we denote by  $\bar{\xi}$  the function on  $\Omega$  which is given by pointwise complex conjugation of  $\xi$ , i.e.,  $\bar{\xi}(\omega) = \overline{\xi(\omega)}$ , for  $\omega \in \Omega$ . The map  $\bar{\square} : L^p(\Omega, \nu) \rightarrow L^p(\Omega, \nu)$  given by  $\xi \mapsto \bar{\xi}$  defines a conjugate linear isometry on  $L^p(\Omega, \nu)$ . Let  $\Phi_p : L^p(\Omega, \nu) \rightarrow \overline{L^p(\Omega, \nu)}$  be the conjugate linear isometry which acts as the identity map on the underlying set. The composition  $\Phi_p \circ \bar{\square} : L^p(\Omega, \nu) \rightarrow \overline{L^p(\Omega, \nu)}$  is a linear isometric isomorphism. It follows that the conjugate of an  $L^p$ -space is itself an  $L^p$ -space. We record this fact in Proposition 1.2.10.

**Proposition 1.2.10.** *The class of  $L^p$ -spaces is closed under conjugation.*

We can now treat symmetrization of the  $L^p$ - and  $p$ -pseudofunction algebras.

**Example 1.2.11.** As an immediate corollary to Proposition 1.2.10, we see that the class  $\text{Rep}_{L^p}(G)$  is closed under conjugation. Further, recall that the dual of an  $L^p$ -space is the  $L^{p'}$ -space on the same measure space. Invoking Example 1.2.9, it follows that the  $L^p$ -pseudofunction algebra can be symmetrized and that the norm on  $F_{L^p}^*(G)$  is given by, for  $f \in L^1(G)$ ,

$$\|f\|_{F_{L^p}^*(G)} = \max\{\|f\|_{F_{L^p}}, \|f\|_{F_{L^{p'}}}\}. \quad \circ$$

**Example 1.2.12.** Denote by  $\lambda_p$  the left regular representation of  $G$  on  $L^p(G)$ . For  $\xi \in L^p(G)$  and  $s, t \in G$ , we have

$$\overline{\lambda_p(s)\Phi_p(\bar{\xi})(t)} = \Phi_p(\bar{\xi})(s^{-1}t) = \overline{\xi(s^{-1}t)} = \Phi_p(\overline{\lambda_p(s)\xi})(t).$$

Hence,  $\Phi_p \circ \bar{\square} : L^p(G) \rightarrow \overline{L^p(G)}$  is an intertwiner of  $\lambda_p$  and its conjugate representation  $\overline{\lambda_p}$ . This shows that  $\lambda_p$  and  $\overline{\lambda_p}$  are equivalent, and so, the set  $\{\lambda_p\}$  is closed under conjugation. We observe further that the dual of  $\lambda_p$  is  $\lambda_{p'}$ . We can then invoke Example 1.2.9 to see that the  $p$ -pseudofunction algebra can be symmetrized. The norm on  $F_{\lambda_p}^*(G)$  is given by, for  $f \in L^1(G)$ ,

$$\|f\|_{F_{\lambda_p}^*(G)} = \max\{\|\lambda_p(f)\|, \|\lambda_{p'}(f)\|\}.$$

Observe that the symmetrization of the  $p$ -pseudofunction algebra introduced here agrees with the *symmetrized  $p$ -pseudofunction algebra* studied in, e.g., [LY17], [SW20] and [SW24].  $\circ$

We turn our attention to the class  $QSL^p$  consisting of Banach spaces which are isometrically isomorphic to a quotient (by a closed subspace) of a closed subspace of an  $L^p$ -space.

**Proposition 1.2.13.** *Let  $V$  be a closed subspace of an  $L^p$ -space and let  $W$  be a closed subspace of  $V$ . Then  $\overline{V/W}$  and  $\overline{\overline{V}/\overline{W}}$  are canonically isometrically isomorphic. In particular, the conjugate of a  $QSL^p$ -space is again a  $QSL^p$ -space.*

*Proof.* Denote by  $\Phi_{V/W} : V/W \rightarrow \overline{V/W}$  and  $\Phi_V : V \rightarrow \overline{V}$  the conjugate linear isometries which act as the identity on the underlying sets. Note that  $\Phi_V$  is the restriction of  $\Phi_p$ . Let  $\Psi : \overline{V/W} \rightarrow \overline{\overline{V}/\overline{W}}$  be the map given by, for  $\xi \in V$ ,

$$\Phi_{V/W}(\xi + W) \mapsto \Phi_V(\xi) + \overline{W}.$$

This is a well-defined map as  $\Phi_V(\xi) \in \overline{W}$  whenever  $\xi \in W$ . To see that it is linear, let  $\xi, \zeta \in V$  and  $\alpha \in \mathbb{C}$ . Then

$$\begin{aligned} \Psi\left(\Phi_{V/W}(\xi + W) + \Phi_{V/W}(\zeta + W)\right) &= \Psi\left(\Phi_{V/W}((\xi + \zeta) + W)\right) \\ &= \Phi_V(\xi + \zeta) + \overline{W} \\ &= \left(\Phi_V(\xi) + \overline{W}\right) + \left(\Phi_V(\zeta) + \overline{W}\right) \\ &= \Psi\left(\Phi_{V/W}(\xi + W)\right) + \Psi\left(\Phi_{V/W}(\zeta + W)\right), \end{aligned}$$

and

$$\begin{aligned} \Psi\left(\alpha \cdot \Phi_{V/W}(\xi + W)\right) &= \Psi\left(\Phi_{V/W}(\overline{\alpha}\xi + W)\right) \\ &= \Phi_V(\overline{\alpha}\xi) + \overline{W} \\ &= \alpha \cdot \Phi_V(\xi) + \overline{W} \\ &= \alpha(\Phi_V(\xi) + \overline{W}) \\ &= \alpha\Psi\left(\Phi_{V/W}(\xi + W)\right). \end{aligned}$$

Finally, to see that  $\Psi$  is isometric, let  $\xi \in V$ . We have,

$$\begin{aligned} \left\|\Psi\left(\Phi_{V/W}(\xi + W)\right)\right\|_{\overline{V}/\overline{W}} &= \left\|\Phi_V(\xi) + \overline{W}\right\|_{\overline{V}/\overline{W}} \\ &= \inf_{\zeta \in W} \left\|\Phi_V(\xi) + \Phi_V(\zeta)\right\|_{\overline{V}} \\ &= \inf_{\zeta \in W} \left\|\Phi_V(\xi + \zeta)\right\|_{\overline{V}} \\ &= \inf_{\zeta \in W} \left\|\xi + \zeta\right\|_V \\ &= \left\|\xi + W\right\|_{V/W} \\ &= \left\|\Phi_{V/W}(\xi + W)\right\|_{\overline{V}/\overline{W}} \end{aligned}$$

If  $E$  is a  $QSL^p$ -space, we can find an  $L^p$ -space  $L^p(\Omega, \nu)$  and closed subspaces  $W \subset V \subset L^p(\Omega, \nu)$  such that  $E$  is isometrically isomorphic to  $V/W$ . By the above,  $\overline{E}$  is isometrically isomorphic to  $\overline{V}/\overline{W}$ , which is a quotient of a subspace of  $\overline{L^p(\Omega, \nu)}$ . Since, by Proposition 1.2.10,  $\overline{L^p(\Omega, \nu)}$  is an  $L^p$ -space, it follows that  $\overline{E}$  is a  $QSL^p$ -space.  $\square$

For a Banach space  $E$  and a subspace  $V \subset E$ , the *annihilator of  $V$  in  $E$* , denoted by  $V^\perp$ , is the subspace of  $E'$  given by

$$V^\perp = \{ \eta \in E' \mid \eta(\xi) = 0, \forall \xi \in V \}.$$

When it is clear from context, we shall refer to  $V^\perp$  simply as the *annihilator of  $V$*  and drop the description “in  $E$ ”. On the other hand, when the context demands it, we will add the ambient space explicitly to the notation as so:  $V^{\perp in E}$ .

Lemma 1.2.14 below is a well-known consequence of the Hahn-Banach theorem. See, e.g., Theorems 1.10.16 and 17 in [Meg98].

**Lemma 1.2.14.** *Let  $E$  be a Banach space and  $V \subset E$  a closed subspace. Then*

$$(a) \ E'/V^\perp \cong V',$$

$$(b) \ (E/V)' \cong V^\perp.$$

**Proposition 1.2.15.** *Let  $E$  be a Banach space and let  $W \subset V \subset E$  be closed subspaces. Then  $V^\perp \subset W^\perp \subset E'$  are closed subspaces and*

$$(V/W)' \cong W^\perp/V^\perp.$$

*Proof.* The annihilators  $V^\perp$  and  $W^\perp$  are closed subspaces of  $E'$ , by Proposition 1.10.15(a) in [Meg98]. The inclusions  $V^\perp \subset W^\perp \subset E'$  are by construction. We have, by Lemma 1.2.14

$$\begin{aligned} (V/W)' &\cong W^{\perp \text{ in } V} = \{ \zeta \in V' \mid \zeta(\xi) = 0, \forall \xi \in W \} \\ &\cong \left\{ \eta + V^\perp \in E'/V^\perp \mid \eta|_V(\xi) = 0, \forall \xi \in W \right\}, \end{aligned}$$

where we have used part (b) for the first isometric isomorphism and part (a) for the second. Now, the latter is a subspace of the quotient  $E'/V^\perp$  consisting of those equivalence classes that are represented by a functional  $\eta$  in the annihilator of  $W$  in  $E$ . Hence,  $(V/W)' \cong W^\perp/V^\perp$ , as we wanted to show.  $\square$

We get the following as an immediate corollary to Proposition 1.2.15.

**Corollary 1.2.16.** *The dual of a  $QSL^p$ -space is a  $QSL^{p'}$ -space.*

**Example 1.2.17.** It follows from Proposition 1.2.13 that the class  $\text{Rep}_{QSL^p}(G)$  is closed under conjugation. Further, by Corollary 1.2.16, we see that the dual of  $\text{Rep}_{QSL^p}(G)$  is  $\text{Rep}_{QSL^{p'}}(G)$ , with  $p'$  the Hölder conjugate of  $p$ . Invoking Example 1.2.9, it follows that the  $QSL^p$ -pseudofunction algebra can be symmetrized and that the norm on  $F_{QSL^p}^*(G)$  is given by, for  $f \in L^1(G)$ ,

$$\|f\|_{F_{QSL^p}^*(G)} = \max\{\|f\|_{F_{QSL^p}}, \|f\|_{F_{QSL^{p'}}}\}. \quad \circ$$

*Remark 1.2.18.* Beware that the symmetrization of a pseudofunction algebra constructed in paper IV differs from the one presented here. Let  $\pi$  be an isometric representation of a locally compact group  $G$  on a reflexive Banach space  $E$ , and assume that  $\pi'$  is strongly continuous so that it is also an isometric representation. In paper IV, the *symmetrized  $\pi$ -pseudofunction algebra* is defined as the completion of  $L^1(G)$  with respect to the norm given by, for  $f \in L^1(G)$ ,

$$\|f\|_{F_{\pi}^*(G)} = \max\{\|f\|_{F_{\pi}}, \|f\|_{F_{\pi'}}\}.$$

The assumption that  $E$  is reflexive ensures that  $\{\pi, \pi'\}$  is closed under duality. But it need not be closed under conjugation. The involution on  $L^1(G)$  need not extend to the symmetrized  $\pi$ -pseudofunction algebra of paper IV. The symmetrization presented in this preliminary section of the thesis is therefore preferable when the intention of the symmetrization is to construct a Banach \*-algebra. On the other hand, the construction in paper IV is better suited for the formulation of Theorem IV.4.3. As we have seen in the examples 1.2.11, 1.2.12 and 1.2.17, the two constructions coincides in the examples most relevant for this thesis.

### 1.3 Spaces of matrix coefficients

Let  $G$  be a locally compact group and let  $(\pi, E)$  be an isometric representation of  $G$  on a Banach space  $E$ . A *matrix coefficient* of  $\pi$  is a complex-valued function on  $G$  of the form

$$\pi_{\xi, \eta}(s) = \eta(\pi(s)\xi), \quad (1.2)$$

for  $s \in G$ , where  $\xi \in E$  and  $\eta \in E'$ . In this section, we discuss spaces of matrix coefficients and how these provide a picture which is dual to the one provided by the pseudofunction algebra construction. Inspired by [CF84], we associate to  $\pi$  the  $\pi$ -Fourier-Stieltjes space  $B_\pi(G)$ . We shall see that  $B_\pi(G)$  embeds contractively inside  $L^\infty(G)$  and that it can be identified with the Banach space dual of the  $\pi$ -pseudofunction algebra  $F_\pi(G)$ . Further, we shall acquaint ourselves with the  $p$ -Fourier-Stieltjes algebra  $B_p(G)$  consisting of matrix coefficients of isometric representations on  $QSL^p$ -spaces. This algebra was introduced by Runde in [Run04] as a  $p$ -analog of the Fourier-Stieltjes algebra which Eymard introduced in [Eym64]. We omit proofs of most statements presented in this section as they are included in paper IV.

Given a measurable function  $\varphi : G \rightarrow \mathbb{C}$ , we may consider this a (not necessarily bounded) linear functional on  $L^1(G)$  by setting, for  $f \in L^1(G)$ ,

$$\varphi(f) = \int_G f(s)\varphi(s) \, d\mu_G(s). \quad (1.3)$$

A subset  $A \subset G$  is called *locally null* if  $\mu_G(B \cap F) = 0$ , for all measurable subsets  $F \subset G$  with  $\mu_G(F) < \infty$ . Trivially, every null set is locally null. The converse need not be true, but it does hold when  $\mu_G$  is semifinite, i.e., when, for every measurable  $A \subset G$  with  $\mu_G(A) = \infty$ , there exists  $F \subset A$  with  $0 < \mu_G(F) < \infty$ . Two measurable functions  $\varphi$  and  $\psi$  on  $G$  are said to be equal *locally almost everywhere* if they agree everywhere except on a locally null set. Observe that two functions which are equal locally almost everywhere define the same functional on  $L^1(G)$ . For the isometric representation  $\pi$ , consider the linear space

$$\mathcal{B}_\pi(G) = \left\{ \varphi : G \rightarrow \mathbb{C} \text{ measurable} \mid \exists C > 0 : |\varphi(f)| \leq C \|\pi(f)\|, \forall f \in L^1(G) \right\}.$$

**Definition 1.3.1.** The  $\pi$ -Fourier-Stieltjes space  $B_\pi(G)$  is the quotient of  $\mathcal{B}_\pi(G)$  where two functions are identified if they are equal locally almost everywhere.

We equip  $\mathcal{B}_\pi(G)$  with the seminorm given by, for  $\varphi \in \mathcal{B}_\pi(G)$ ,

$$\|\varphi\|_{\mathcal{B}_\pi} = \inf \left\{ C > 0 \mid |\varphi(f)| \leq C \|\pi(f)\|, \forall f \in L^1(G) \right\}.$$

Then  $\|\varphi\|_{\mathcal{B}_\pi} = 0$  exactly when  $\varphi$  is locally almost everywhere zero. Hence,  $\|\cdot\|_{\mathcal{B}_\pi}$  defines a norm on  $B_\pi(G)$ .

Denote by  $\mathcal{L}^\infty(G)$  the space of bounded measurable complex-valued functions on  $G$ . We define a seminorm on  $\mathcal{L}^\infty(G)$  by setting, for  $\varphi \in \mathcal{L}^\infty(G)$ ,

$$\|\varphi\|_\infty^* = \inf \left\{ M > 0 \mid \{s \in G : |\varphi(s)| > M\} \text{ is locally null} \right\}.$$

We let  $L^\infty(G)$  denote the quotient of  $\mathcal{L}^\infty(G)$  with respect to the equivalence relation identifying functions that are equal locally almost everywhere. Then  $L^\infty(G)$  is a Banach space when equipped with the norm  $\|\cdot\|_\infty^*$ .

*Remark 1.3.2.* Recall that  $L^\infty(G)$  is usually constructed as the quotient of  $\mathcal{L}^\infty(G)$  where two functions are identified if they are equal *almost everywhere*. The usual construction, however, may give an algebra which is too big for our purposes. Indeed, in both Proposition 1.3.3 and 1.3.4, it is imperative to use the version of  $L^\infty(G)$  where functions are identified *locally almost everywhere*. The convention to use  $L^\infty(G)$  to denote the quotient modulo equality locally almost everywhere is taken from appendix I in [Wil07]. Since we shall not need the usual construction of  $L^\infty(G)$ , the ambiguity in notation caused by this convention will not cause any confusion. When  $G$  is  $\sigma$ -compact, its Haar measure is automatically semifinite, and so,  $L^\infty(G)$ , as it is constructed here, agrees with its usual definition.

**Proposition 1.3.3.** *The  $\pi$ -Fourier-Stieltjes space  $B_\pi(G)$  embeds canonically and contractively into  $L^\infty(G)$ .*

*Proof.* Let  $\varphi \in B_\pi(G)$ . For each  $M > 0$ , set  $A_M = \{s \in G \mid |\varphi(s)| > M\}$ . For a given  $M > 0$ , if  $A_M$  is not locally null, we can find a measurable set  $F \subset G$  with  $\mu_G(F) < \infty$  such that  $\mu_G(A_M \cap F) > 0$ . Set

$$f_M = \overline{\text{sign}(\varphi)} 1_{A_M \cap F}$$

Then  $f_M$  is integrable and

$$\varphi(f_M) = \int_{A_M \cap F} |\varphi| \, d\mu_G > M \mu_G(A_M \cap F) = M \|f_M\|_1 \geq M \|\pi(f_M)\|.$$

By contraposition, if  $M > 0$  is such that  $|\varphi(f)| \leq M \|\pi(f)\|$ , for all  $f \in L^1(G)$ , then  $A_M$  must be locally null. Hence  $\|\varphi\|_\infty^* \leq \|\varphi\|_{B_\pi}$ . Hence, the canonical map sending  $\varphi \in B_\pi(G)$  to its equivalence class in  $L^\infty(G)$  is a contraction.  $\square$

When  $\varphi \in \mathcal{L}^\infty(G)$  is bounded as a function on  $G$ , it is also bounded when considered a linear functional on  $L^1(G)$  via equation (1.3). We obtain hereby a well-defined linear map  $L^\infty(G) \rightarrow L^1(G)'$  which we refer to as the canonical map between these spaces. By Proposition I.27 in [Wil07], this canonical map identifies  $L^\infty(G)$  and  $L^1(G)'$ . We restate this fact without proof in Proposition 1.3.4.

**Proposition 1.3.4.** *The canonical map  $L^\infty(G) \rightarrow L^1(G)'$  defined in equation (1.3) is an isometric isomorphism.*

Recall that the  $\pi$ -pseudofunction algebra  $F_\pi(G)$  was constructed as a completion of (an appropriate quotient of)  $L^1(G)$ , and so, its Banach space dual  $F_\pi(G)'$  embeds contractively into the dual of  $L^1(G)$ . On the other hand, we saw in Proposition 1.3.3 that  $B_\pi(G)$  embeds contractively into  $L^\infty(G)$ . Proposition 1.3.5, which is Proposition 4.1 in Paper IV, establishes that the canonical identification of  $L^\infty(G)$  with  $L^1(G)'$  restricts to an identification of  $B_\pi(G)$  with  $F_\pi(G)'$ .

**Proposition 1.3.5.** *The restriction to  $B_\pi(G)$  of the map  $L^\infty(G) \rightarrow L^1(G)'$  defined in equation (1.3) defines an isometric isomorphism  $B_\pi(G) \rightarrow F_\pi(G)'$ .*

Given  $\xi \in E$  and  $\eta \in E'$ , the matrix coefficient  $\pi_{\xi,\eta}$  is continuous because  $\pi$  is strongly continuous by assumption. Further, we have  $\|\pi_{\xi,\eta}\|_{B_\pi} \leq \|\xi\| \|\eta\|$ . Hence,  $B_\pi(G)$  contains all matrix coefficients for  $\pi$ . However, it is not a priori clear that the matrix coefficients for  $\pi$  exhaust  $B_\pi(G)$ . In Theorem 1.3.6, which is Theorem 2 in [CF84], we make precise in what way  $B_\pi(G)$  can still be understood as a space of matrix coefficients.

**Theorem 1.3.6.** *Let  $\pi$  be an isometric representation of  $G$  on a Banach space  $E$ . There exists an isometric representation  $\pi_0$  on a Banach space  $E_0$  such that  $B_\pi(G)$  and  $B_{\pi_0}(G)$  are canonically isometrically isomorphic and such that, for every  $\varphi \in B_\pi(G)$ , one can find  $\xi \in E_0$  and  $\eta \in E'_0$  such that  $\varphi(s) = \eta(\pi_0(s)\xi)$ , for  $s \in G$ , and  $\|\varphi\|_{B_\pi} = \|\xi\| \|\eta\|$ .*

The Fourier-Stieltjes algebra of  $G$ , denoted  $B(G)$ , was introduced by Eymard in [Eym64]. It is the collection of matrix coefficients for unitary representations on some Hilbert space. In [Run04], Runde introduced a  $p$ -analog of this algebra as follows: For  $1 < p < \infty$ , denote by  $QSL^p$  the class of Banach spaces isometrically isomorphic to a quotient of a subspace of an  $L^p$ -space, and by  $\text{Rep}_p(G)$  the class of isometric representations of  $G$  on a space in  $QSL^p$ .

**Definition 1.3.7.** The  $p$ -Fourier-Stieltjes algebra, for a fixed  $1 < p < \infty$ , is the collection of matrix coefficients for representations in  $\text{Rep}_p(G)$ :

$$B_p(G) = \left\{ \pi_{\xi,\eta} \mid (\pi, E) \in \text{Rep}_p(G), \xi \in E, \eta \in E' \right\}.$$

It is shown in [Run04] that  $B_p(G)$  is a commutative Banach algebra over  $\mathbb{C}$  with pointwise operations and when equipped with the norm given by, for  $\varphi \in B_p(G)$ :

$$\|\varphi\|_{B_p} = \inf \left\{ \|\xi\| \|\eta\| \mid \varphi = \pi_{\xi,\eta}, \text{ for } (\pi, E) \in \text{Rep}_p(G), \xi \in E, \eta \in E' \right\}.$$

**Warning!** In [Run04], Runde defines  $B_p(G)$  as the collection of matrix coefficients for representations in  $\text{Rep}_{p'}(G)$ , where  $p'$  is the Hölder conjugate of  $p$ . We follow instead the convention of [Daw10].

*Remark 1.3.8.* For  $p = 2$ , the 2-Fourier-Stieltjes algebra  $B_2(G)$  recovers the Fourier-Stieltjes algebra of Eymard.

Theorem 1.3.9, which is Theorem 2.4 in paper IV, shows how the  $p$ -Fourier-Stieltjes algebra relates to the  $\pi$ -Fourier-Stieltjes spaces of isometric representations on  $QSL^p$ -spaces, and that it is itself a special case of this more general construction.

**Theorem 1.3.9.** *Let  $1 \leq p < \infty$  and let  $(\pi, E)$  be an isometric representation of the locally compact group  $G$  on a  $QSL^p$ -space. Then  $B_\pi(G)$  embeds canonically and contractively into  $B_p(G)$ . Moreover, there exists an isometric representation on a  $QSL^p$ -space such that this embedding is an isometric isomorphism.*

*Remark 1.3.10.* The second part of Theorem 1.3.9 establishes that  $B_p(G)$  can be realized as the space  $B_{\pi_0}(G)$ , for some isometric representation  $\pi_0$  on a  $QSL^p$ -space. For this purpose, one can take  $\pi_0$  to be  $p$ -universal in the sense that it contains all cyclic representations on  $QSL^p$ -spaces. The existence of such a representation is explained in [Run04], and we recap this in paper IV.

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## Chapter 2

# Isometric representations on $L^p$ -spaces

When  $p \neq 2$ , the group of surjective isometries on an  $L^p$ -space on a  $\sigma$ -finite measure space is described by the Banach-Lamperti theorem. The theorem was announced without proof in the special case of the unit interval with the Lebesgue measure in the monograph [Ban32] by Banach. It was later generalized and proven by Lamperti in [Lam58]. We give in Sections 2.1 and 2.3 the necessary background before we present the result of Banach and Lamperti in Theorem 2.3.4. A modern account of the Banach-Lamperti theorem in the more general setting of localizable measure algebras can be found in [Gar21]. In Section 2.4, we apply the Banach-Lamperti theorem to describe the isometric representation theory of a locally compact group on  $L^p$ -spaces on  $\sigma$ -finite measure spaces when  $p \neq 2$ . Understanding the representation theory on  $L^p$ -spaces when  $p \neq 2$  will become important to us in paper III and IV.

### 2.1 Isometries of $(\mathbb{R}^2, \|\cdot\|_p)$ – A toy example

Consider the plane equipped with the  $p$ -norm. We can think of this either as the space  $(\mathbb{R}^2, \|\cdot\|_p)$ , or as the real  $L^p$ -space  $\ell^p(\{1, 2\}; \mathbb{R})$  where  $\{1, 2\}$  is equipped with the counting measure. Our aim in this section is to characterize its isometries when  $p \neq 2$  and hereby prove the Banach-Lamperti theorem in this special case. Observe that the isometries of  $(\mathbb{R}^2, \|\cdot\|_p)$  are automatically surjective because the space is finite dimensional. It is pointed out in [Gar21] that the special case of  $(\mathbb{R}^2, \|\cdot\|_p)$  is suitable for illustrating the Banach-Lamperti theorem; it lends itself to illustration as functions allow themselves to be drawn as points on a piece of paper. We include it here to showcase the core idea of the proof of the Banach-Lamperti theorem as well as to elucidate an important difference in this context between when  $p$  equals 2 and when it does not: On an  $L^2$ -space, there is an abundance of surjective isometries. The situation simplifies greatly when  $p \neq 2$ .

The isometries on  $(\mathbb{R}^2, \|\cdot\|_p)$  are full rank  $2 \times 2$ -matrices which preserve the unit sphere, the shape of which depends on the value of  $p$ . In Figure 1, we have drawn the unit spheres for different values of  $p$ . A first glance on their shapes

already gives a good idea of why  $p = 2$  should be thought of as a case separate from the rest: For  $p = 2$ , the unit sphere is a circle, and its shape is thus preserved by rotations of the plane by any angle. By contrast, the only rotations that will preserve the  $p$ -unit sphere when  $p \neq 2$  are by multiples of  $90^\circ$ . The circle has more symmetries than the squarish shapes that are the  $p$ -unit spheres when  $p \neq 2$ . This is the picture we have in mind when we say that  $L^2$ -spaces admit an abundance of surjective isometries while the situation simplifies for  $p \neq 2$ .

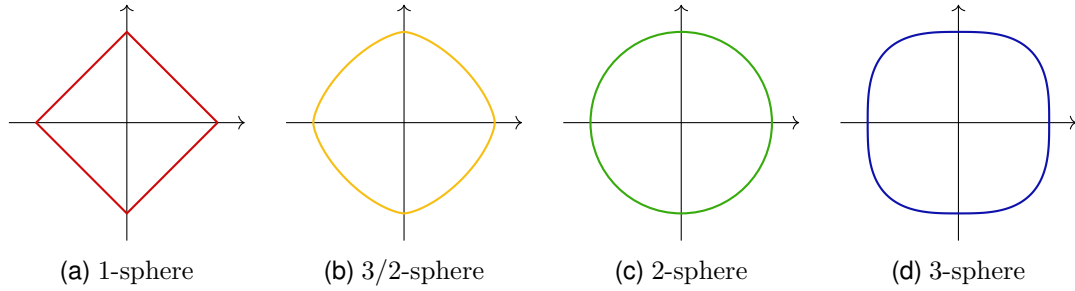


Figure 1: Unit spheres of  $(\mathbb{R}^2, \|\cdot\|_p)$  for different values of  $p$ .

We consider two basic types of isometries on  $(\mathbb{R}^2, \|\cdot\|_p)$ :

**Multiplication by a phase:** The two maps  $m_x : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and  $m_y : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by, for  $(x_1, x_2) \in \mathbb{R}^2$ ,

$$m_x(x_1, x_2) = (x_1, -x_2) \quad \text{and} \quad m_y(x_1, x_2) = (-x_1, x_2),$$

are isometries on  $(\mathbb{R}^2, \|\cdot\|_p)$ .

**Index swap:** The map  $u_\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by, for  $(x_1, x_2) \in \mathbb{R}^2$ ,

$$u_\sigma(x_1, x_2) = (x_2, x_1).$$

is an isometry on  $(\mathbb{R}^2, \|\cdot\|_p)$ .

Geometrically,  $m_x$  is a reflection over the  $x$ -axis,  $m_y$  is a reflection over the  $y$ -axes, and  $u_\sigma$  is a reflection over the line  $y = x$ . The group generated by these three isometries contains, besides the generators themselves, the reflection over the line  $y = -x$  as well as all rotations by multiples of  $90^\circ$ . For example, a counterclockwise rotation of  $90^\circ$  can be performed by applying first  $m_x$  and then  $u_\sigma$ . But it does not contain rotations by any angle which is not a multiple of  $90^\circ$ . We recognize the group generated by  $m_x$ ,  $m_y$  and  $u_\sigma$  as isomorphic to the symmetry group of a square. When  $p \neq 2$ , Theorem 2.1.1 states that  $m_x$ ,  $m_y$  and  $u_\sigma$  generate the whole group of surjective isometries of  $(\mathbb{R}^2, \|\cdot\|_p)$ .

**Theorem 2.1.1.** *Let  $1 \leq p < \infty$ ,  $p \neq 2$ , and let  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be an isometry with respect to the norm  $\|\cdot\|_p$ . Then  $T$  can be decomposed into a composition of the isometries  $m_x$ ,  $m_y$  and  $u_\sigma$ .*

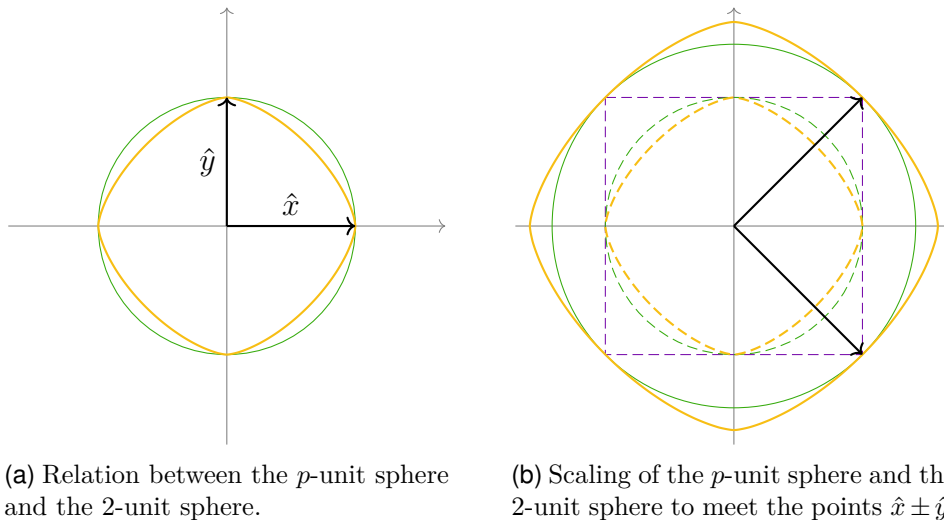
Theorem 2.1.1 is a special case of the Banach-Lamperti theorem, which we shall state and prove in greater generality in Theorem 2.3.4. We sketch the proof in the special case of  $(\mathbb{R}^2, \|\cdot\|_p)$  for illustrative purposes.

## 2.1. Isometries of $(\mathbb{R}^2, \|\cdot\|_p)$ – A toy example

*Sketch of proof of Theorem 2.1.1.* Denote by  $\hat{x} = (1, 0)$  and  $\hat{y} = (0, 1)$  the standard basis vectors for  $\mathbb{R}^2$ . Consider the four vectors  $T(\hat{x})$ ,  $T(\hat{y})$ ,  $T(\hat{x} + \hat{y})$  and  $T(\hat{x} - \hat{y})$ . Because  $T$  is an isometry with respect to  $\|\cdot\|_p$ , we must have  $\|T(\hat{x})\|_p = \|T(\hat{y})\|_p = 1$  and  $\|T(\hat{x} \pm \hat{y})\|_p = \|\hat{x} \pm \hat{y}\|_p = 2^{1/p}$ . The idea is to compare the  $p$ -norm of these four vectors with their 2-norm. This will allow us to make use of the parallelogram identity for the inner product space  $(\mathbb{R}^2, \|\cdot\|_2)$ :

$$\|T(\hat{x} + \hat{y})\|_2^2 + \|T(\hat{x} - \hat{y})\|_2^2 = 2 \left( \|T(\hat{x})\|_2^2 + \|T(\hat{y})\|_2^2 \right). \quad (2.1)$$

Assume that  $p < 2$ . In this case, the  $p$ -unit sphere lies *inside* the 2-unit sphere. We illustrate this in Figure 2a. When we scale the  $p$ -unit sphere by a factor of  $2^{1/p}$  and the 2-unit sphere by a factor of  $\sqrt{2}$  so that they meet the points  $\hat{x} \pm \hat{y}$ , the situation reverses: The scaled  $p$ -unit sphere lies *outside* the scaled 2-unit sphere. This is illustrated in Figure 2b.



**Figure 2:** The  $p$ -unit sphere and the 2-unit sphere when  $p < 2$ .

From these observations, we get the following bounds on the 2-norm of the vectors  $T(\hat{x})$ ,  $T(\hat{y})$  and  $T(\hat{x} \pm \hat{y})$ :

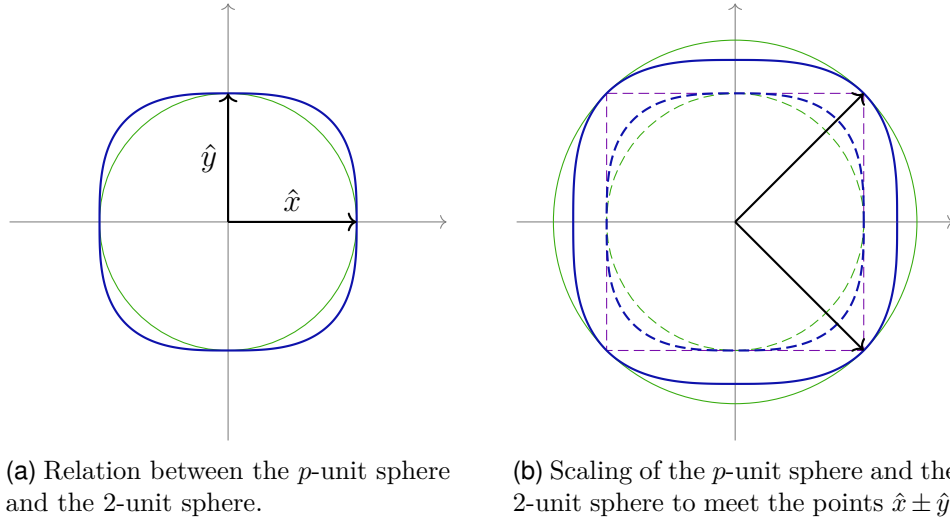
$$\|T(\hat{x})\|_2 \leq 1, \quad \|T(\hat{y})\|_2 \leq 1 \quad \text{and} \quad \|T(\hat{x} \pm \hat{y})\|_2 \geq \sqrt{2}.$$

This in turn gives us the following bounds on the left- and right-hand sides of equation (2.1):

$$4 \leq \|T(\hat{x} + \hat{y})\|_2^2 + \|T(\hat{x} - \hat{y})\|_2^2 = 2 \left( \|T(\hat{x})\|_2^2 + \|T(\hat{y})\|_2^2 \right) \leq 4.$$

It follows that  $\|T(\hat{x})\|_2 = \|T(\hat{y})\|_2 = 1$ , and so,  $T(\hat{x})$  and  $T(\hat{y})$  must lie in the intersection of the  $p$ -unit sphere and the 2-unit sphere. This leaves only 4 possibilities:  $\pm\hat{x}$  and  $\pm\hat{y}$ . From here, it is easy to see that  $T$  must be a multiple of  $m_x$ ,  $m_y$  and  $u_\sigma$ .

The argument in the case  $p > 2$  is analogous except in this case the inequalities are reversed. Indeed, for  $p > 2$ , the  $p$ -unit sphere lies *outside* the 2-unit sphere. And when we scale the  $p$ -unit sphere and the 2-unit sphere to meet the points  $\hat{x} \pm \hat{y}$ , the scaled  $p$ -unit sphere lies *inside* the scaled 2-unit sphere. This is illustrated in Figure 3.  $\square$


 Figure 3: The  $p$ -unit sphere and the 2-unit sphere when  $p > 2$ .

## 2.2 Automorphisms of measure spaces

We provide in this section the background on automorphisms of measure spaces necessary to state and prove the Banach-Lamperti theorem. We define and discuss homomorphisms of measure spaces, which are dubbed *regular set isomorphisms* by Lamperti in [Lam58]. The material presented here can also be found in [Gar21].

Let  $(\Omega, \Sigma, \nu)$  be a measure space. For  $A, B \in \Sigma$ , we write  $A \sim B$  if  $\nu(A \Delta B) = 0$ . Then  $\sim$  defines an equivalence relation on  $\Sigma$ .

**Definition 2.2.1.** A *homomorphism* of  $(\Sigma, [\nu])$  is a map  $\sigma : \Sigma \rightarrow \Sigma$  satisfying the following conditions:

$$\sigma(A) \sim \sigma(B) \text{ if and only if } A \sim B, \text{ for all } A, B \in \Sigma, \quad (2.2)$$

$$\sigma(\Omega) \sim \Omega, \quad (2.3)$$

$$\sigma(A^c) \sim \sigma(A)^c, \text{ for all } A \in \Sigma, \quad (2.4)$$

$$\sigma\left(\bigcup_{n=1}^{\infty} A_n\right) \sim \bigcup_{n=1}^{\infty} \sigma(A_n), \text{ for } (A_n)_{n \geq 1} \text{ pairwise disjoint.} \quad (2.5)$$

A homomorphism  $\sigma : \Sigma \rightarrow \Sigma$  is an *isomorphism* if it furthermore satisfies:

$$\text{For every } B \in \Sigma, \text{ there exists } A \in \Sigma \text{ for which } \sigma(A) \sim B. \quad (2.6)$$

We denote by  $\text{Hom}(\Sigma, [\nu])$  the set of all homomorphisms of  $(\Sigma, [\nu])$  and by  $\text{Aut}(\Sigma, [\nu])$  the set of all isomorphisms of  $(\Sigma, [\nu])$ .

*Remark 2.2.2.* Let  $\sigma \in \text{Hom}(\Sigma, [\nu])$ . Condition (2.2) of Definition 2.2.1 ensures that  $\sigma$  can be considered a well-defined and injective map on the quotient  $\Sigma/\sim$ . As such,  $\sigma$  is surjective exactly when it satisfies condition (2.6). The set  $\text{Aut}(\Sigma, [\nu])$  is therefore a group with product given by composition when its elements are viewed as maps on  $\Sigma/\sim$ .

We collect in Proposition 2.2.3 a few useful properties of homomorphisms of  $(\Sigma, [\nu])$ . These are straightforward to verify.

**Proposition 2.2.3.** *Let  $\sigma \in \text{Hom}(\Sigma, [\nu])$ , let  $A, B \in \Sigma$  and let  $(A_n)_{n \geq 1}$  be a countable family of sets in  $\Sigma$ .*

- (i)  $\sigma(\emptyset) \sim \emptyset$ ,
- (ii)  $\sigma(B \setminus A) \sim \sigma(B) \setminus \sigma(A)$ , for  $A \subset B$ ,
- (iii)  $\sigma(\bigcup_{n=1}^{\infty} A_n) \sim \bigcup_{n=1}^{\infty} \sigma(A_n)$ ,
- (iv)  $\sigma(\bigcap_{n=1}^{\infty} A_n) \sim \bigcap_{n=1}^{\infty} \sigma(A_n)$ ,
- (v)  $\sigma(A \Delta B) \sim \sigma(A) \Delta \sigma(B)$ .

A map  $\sigma : \Sigma \rightarrow \Sigma$  is called *measure preserving* if  $\nu(\sigma(A)) = \nu(A)$ , for all  $A \in \Sigma$ . Further, it is called *measure class preserving* if

$$\nu(\sigma(A)) = 0 \text{ if and only if } \nu(A) = 0, \text{ for all } A \in \Sigma.$$

While homomorphisms of  $(\Sigma, [\nu])$  need not be measure preserving, they are always measure class preserving. This is easy to see when inserting the empty set in place of  $B$  in condition (2.2) of Definition 2.2.1. We record this fact in Proposition 2.2.4. It is the reason for the square-bracket around the measure in  $\text{Hom}(\Sigma, [\nu])$ , respectively,  $\text{Aut}(\Sigma, [\nu])$ . The subsets of measure preserving homomorphisms, respectively, isomorphisms are denoted by  $\text{Hom}(\Sigma, \nu)$ , respectively,  $\text{Aut}(\Sigma, \nu)$ .

**Proposition 2.2.4.** *Let  $\sigma \in \text{Hom}(\Sigma, [\nu])$ . Then  $\sigma$  is measure class preserving.*

One important example of isomorphisms of  $(\Sigma, [\nu])$  derives from transformations of the measure space  $\Omega$ . We elaborate on this example, which is guiding for our intuition, in Example 2.2.5 below.

**Example 2.2.5.** A *bi-measurable transformation* of  $\Omega$  is a bijection on  $\Omega$  such that both itself and its inverse are measurable maps. Given a bi-measurable transformation  $\sigma$  of  $\Omega$ , we denote by  $\sigma_*\nu$  the *push forward measure* of  $\nu$  by  $\sigma$ , i.e., the measure given by  $\sigma_*\nu(A) = \nu(\sigma^{-1}(A))$ , for measurable subsets  $A \subset \Omega$ . We say that  $\nu$  is *invariant* with respect to  $\sigma$  if  $\sigma_*\nu = \nu$ , and we say that  $\nu$  is *quasi-invariant* with respect to  $\sigma$  if  $\sigma_*\nu$  is equivalent to  $\nu$  in the sense that they have the same null-sets. We denote by  $\text{Aut}(\Omega, \nu)$  the collection of all bi-measurable transformations of  $\Omega$  that leave  $\nu$  invariant, and by  $\text{Aut}(\Omega, [\nu])$  the collection of those that leave  $\nu$  quasi-invariant. Each  $\sigma \in \text{Aut}(\Omega, [\nu])$  yields an isomorphism of  $(\Sigma, [\nu])$  by mapping  $A \in \Sigma$  to its image  $\sigma(A)$ . We shall adopt the symbol overload suggested by the notation and use the symbol  $\sigma$  both for the measure space automorphism and the resulting isomorphism of  $(\Sigma, [\nu])$ . In many cases we care about, there is also a converse. For example, when  $\Omega$  is a standard Borel space equipped with a Radon measure  $\nu$ , it is shown in [Neu32] that any measure preserving isomorphism of  $(\Sigma, [\nu])$  arises as described here from a bi-measurable transformation which leaves  $\nu$  invariant. Hence, in this case, we have  $\text{Aut}(\Sigma, \nu) = \text{Aut}(\Omega, \nu)$ .  $\circ$

Denote by  $L^0(\Omega, \nu)$  the space of complex valued, measurable functions on  $\Omega$  where two functions are identified if they agree  $\nu$ -almost everywhere. For each  $\sigma \in \text{Hom}(\Sigma, [\nu])$ , there is a well defined linear map  $u_{0,\sigma}$  on  $L^0(\Omega, \nu)$  which satisfies, for  $A \in \Sigma$ ,

$$u_{0,\sigma}(1_A) = 1_{\sigma(A)}. \quad (2.7)$$

When  $\sigma \in \text{Aut}(\Omega, [\nu])$ , the associated linear map on  $L^0(\Omega, \nu)$  is given by precomposition with  $\sigma^{-1}$ . With this in mind, we shall often write  $\xi \circ \sigma^{-1}$  rather than  $u_{0,\sigma}(\xi)$ , for  $\xi \in L^0(\Omega, \nu)$ , also when  $\sigma$  is a general homomorphism of  $(\Sigma, [\nu])$ .

Given  $\sigma \in \text{Hom}(\Sigma, [\nu])$ , we denote by  $\Sigma_\sigma$  the collection of all sets in  $\Sigma$  which are similar to a set in the image of  $\sigma$  in the sense of the equivalence relation  $\sim$ . Then  $\Sigma_\sigma$  is a  $\sigma$ -algebra on  $\Omega$  by the conditions (2.3)–(2.5) of Definition 2.2.1. We have  $\Sigma_\sigma \subset \Sigma$  with equality exactly when  $\sigma$  is an isomorphism of  $(\Sigma, [\nu])$ .

For each  $A \in \Sigma_\sigma$ , pick a set  $\sigma^{-1}(A) \in \Sigma$  such that  $\sigma(\sigma^{-1}(A)) \sim A$ . Then  $\sigma_*\nu$  given by  $\sigma_*\nu(A) = \nu(\sigma^{-1}(A))$ , for  $A \in \Sigma_\sigma$ , defines a measure on  $(\Omega, \Sigma_\sigma)$ . We refer to  $\sigma_*\nu$  as the *push forward measure* of  $\nu$  by  $\sigma$ . If  $\sigma$  is an isomorphism of  $(\Sigma, [\nu])$  that stems from an automorphism of the measure space as in Example 2.2.5, the push forward measure of  $\nu$  by  $\sigma$ , as it is defined in Example 2.2.5 with  $\sigma$  considered an automorphism of the measure space, agrees with the push forward measure defined here. Hence, our convention to use the symbol  $\sigma$  both for the measure space automorphism and the resulting isomorphism of  $(\Sigma, [\nu])$  should cause no confusion.

Assume  $(\Omega, \Sigma, \nu)$  is a  $\sigma$ -finite measure space. Then the push forward measure of  $\nu$  by a homomorphism of  $(\Sigma, [\nu])$  is again  $\sigma$ -finite. Further, for  $\sigma \in \text{Hom}(\Sigma, [\nu])$ , it follows from Proposition 2.2.4 that the two measures  $\nu$  and  $\sigma_*\nu$  on  $\Sigma_\sigma$  have the same null-sets. This allows us to apply the Radon-Nikodym theorem (see, e.g., Theorem 3.8 in [Fol99] and Theorem 2.7 in [Gar21]). The Radon-Nikodym derivative  $\frac{d\sigma_*\nu}{d\nu}$  is then a strictly positive and  $\Sigma_\sigma$ -measurable function. Moreover, for  $\sigma, \rho \in \text{Hom}(\Sigma, [\nu])$ , their composition is again a homomorphism of  $(\Sigma, [\nu])$  and the following identities hold:

$$\frac{d(\sigma \circ \rho)_*\nu}{d\nu} = \frac{d\rho_*\sigma_*\nu}{d\nu}, \quad (2.8)$$

$$\frac{d\rho_*\sigma_*\nu}{d\sigma_*\nu} = \frac{d\rho_*\nu}{d\nu} \circ \sigma^{-1}. \quad (2.9)$$

As a direct corollary to the Radon-Nikodym theorem, we get the *change of variable formula*, which plays a central role in our study of isometries on  $L^p$ -spaces. We state it here for easy reference.

**Change of variable formula.** *Let  $(\Omega, \Sigma, \nu)$  be a  $\sigma$ -finite measure space and let  $\sigma \in \text{Hom}(\Sigma, [\nu])$ . Then, for all  $\xi \in L^1(\Omega, \nu)$ ,*

$$\int_{\Omega} \xi \, d\nu = \int_{\Omega} \xi \circ \sigma^{-1} \frac{d\sigma_*\nu}{d\nu} \, d\nu.$$

## 2.3 Surjective isometries on $L^p$ -spaces

Let  $(\Omega, \Sigma, \nu)$  be a  $\sigma$ -finite measure space, let  $1 \leq p < \infty$ , and denote by  $L^p(\Omega, \nu)$  the linear subspace of  $L^0(\Omega, \nu)$  consisting of the (equivalence classes of) functions that are  $p$ -integrable. In this section, we state and prove the Banach-Lamperti theorem, which characterizes the linear surjective isometries on  $L^p(\Omega, \nu)$  when  $p \neq 2$ . We begin our investigation by discussing two natural ways of constructing such maps: By multiplication with a phase or by precomposition with an automorphism. All proofs given in this section can be found in [Lam58] and [GT22] (see also [Gar21]).

**Multiplication with a phase.** Denote by  $L^0(\Omega, \nu; \mathbb{T})$  the subset of  $L^0(\Omega, \nu)$  consisting of the measurable functions with values in the unit circle  $\mathbb{T}$ . The set  $L^0(\Omega, \nu; \mathbb{T})$  comes equipped with a natural group structure with multiplication defined pointwise via the group structure of  $\mathbb{T}$ . The identity is the constant function  $1_\Omega$  and the inverse is given by complex conjugation. For each function  $c \in L^0(\Omega, \nu; \mathbb{T})$ , the associated *multiplier*  $m_c$  is the surjective linear isometry on  $L^p(\Omega, \nu)$  given by, for  $\xi \in L^p(\Omega, \nu)$ ,

$$m_c(\xi) = c \cdot \xi. \quad (2.10)$$

We obtain an injective group homomorphism  $m : L^0(\Omega, \nu; \mathbb{T}) \rightarrow \text{Isom}(L^p(\Omega, \nu))$  by setting  $m(c) = m_c$ .

**Precomposition with an automorphism.** For  $\sigma \in \text{Hom}(\Sigma, [\nu])$ , we have defined in equation (2.7) a linear map  $u_{0,\sigma}$  on  $L^0(\Omega, \nu)$  which we think of as a precomposition with an automorphism. To obtain a linear isometry on  $L^p(\Omega, \nu)$ , we modify  $u_{0,\sigma}$  as follows: Define a linear map  $u_{p,\sigma} : L^p(\Omega, \nu) \rightarrow L^p(\Omega, \nu)$  by, for  $\xi \in L^p(\Omega, \nu)$ ,

$$u_{p,\sigma}\xi = \left( \frac{d\sigma_*\nu}{d\nu} \right)^{1/p} \xi \circ \sigma^{-1}, \quad (2.11)$$

Then  $u_{p,\sigma}$  defines an isometry on  $L^p(\Omega, \nu)$  by the change of variable formula. Further, when  $\sigma \in \text{Aut}(\Sigma, [\nu])$  so that it satisfies condition (2.6) of Definition 2.2.1, we see that  $u_{p,\sigma}$  is also surjective. Indeed, when  $\sigma$  satisfies (2.6) we can define an isomorphism of  $(\Sigma, [\nu])$ ,  $\sigma^{-1} : \Sigma \rightarrow \Sigma$ , such that  $\sigma(\sigma^{-1}(A)) \sim A$ , for all  $A \in \Sigma$ . Then, for  $A \in \Sigma$ ,

$$u_{p,\sigma} \left( \left( \frac{d(\sigma^{-1})_*\nu}{d\nu} \right)^{1/p} 1_{\sigma^{-1}(A)} \right) = \left( \frac{d\sigma_*\nu}{d\nu} \cdot \frac{d(\sigma^{-1})_*\nu}{d\nu} \circ \sigma^{-1} \right)^{1/p} 1_{\sigma(\sigma^{-1}(A))} = 1_A.$$

Hence, all indicator functions on sets from  $\Sigma$  are in the image of  $u_{p,\sigma}$ . It follows that  $u_{p,\sigma}$  is surjective because it is linear and continuous and the simple functions are dense in  $L^p(\Omega, \nu)$ . We obtain an injective map  $u_p : \text{Aut}(\Sigma, [\nu]) \rightarrow \text{Isom}(L^p(\Omega, \nu))$  by setting  $u_p(\sigma) = u_{p,\sigma}$ . It follows from the uniqueness part of the Radon-Nikodym theorem that this is a group homomorphism.

We have now identified two subgroups of  $\text{Isom}(L^p(\Omega, \nu))$ , namely,  $L^0(\Omega, \nu; \mathbb{T})$  and  $\text{Aut}(\Sigma, [\nu])$ . As subgroups, they interrelate nicely with each other by satisfying a covariance relation, which we state precisely in Proposition 2.3.1.

**Proposition 2.3.1.** *Let  $(\Omega, \nu)$  be a  $\sigma$ -finite measure space. Then each pair  $c \in L^0(\Omega, \nu; \mathbb{T})$  and  $\sigma \in \text{Aut}(\Sigma, [\nu])$  satisfy the following covariance relation:*

$$u_{p,\sigma} m_c u_{p,\sigma}^{-1} = m_{c \circ \sigma^{-1}}. \quad (2.12)$$

*Proof.* Let  $c \in L^0(\Omega, \nu; \mathbb{T})$  and  $\sigma \in \text{Aut}(\Sigma, [\nu])$ . For each  $\xi \in L^p(\Omega, \nu)$ , we have

$$\begin{aligned} u_{p,\sigma} m_c u_{p,\sigma}^{-1} \xi &= u_{p,\sigma} \left( c \cdot (\xi \circ \sigma) \cdot \left( \frac{d(\sigma^{-1})_* \nu}{d\nu} \right)^{1/p} \right) \\ &= (c \circ \sigma^{-1}) \cdot (\xi \circ \sigma \circ \sigma^{-1}) \cdot \left( \left( \frac{d(\sigma^{-1})_* \nu}{d\nu} \right)^{1/p} \circ \sigma^{-1} \right) \cdot \left( \frac{d\sigma_* \nu}{d\nu} \right)^{1/p} \\ &= m_{c \circ \sigma^{-1}} \xi \cdot \left( \frac{d(\sigma^{-1})_* \sigma_* \nu}{d\sigma_* \nu} \frac{d\sigma_* \nu}{d\nu} \right)^{1/p} \\ &= m_{c \circ \sigma^{-1}} \xi. \end{aligned}$$

In this computation, we have employed equation (2.9) in the third equality together with the definition of the map  $m : L^0(\Omega, \nu; \mathbb{T}) \rightarrow \text{Isom}(L^p(\Omega, \nu))$  given in equation (2.10). In the fourth equality, we have employed equation (2.8) together with the chain rule and the fact that the Radon-Nikodym derivative of a measure with respect to itself is the constant function 1. Because  $\xi \in L^p(\Omega, \nu)$  was arbitrary, equation (2.12) follows.  $\square$

The semidirect product  $L^0(\Omega, \nu; \mathbb{T}) \rtimes \text{Aut}(\Sigma, [\nu])$ , with respect to the action of  $\text{Aut}(\Sigma, [\nu])$  on  $L^0(\Omega, \nu; \mathbb{T})$  by precomposition as defined in equation (2.7), is the group which as a set is the cartesian product and with multiplication defined by, for  $c, d \in L^0(\Omega, \nu; \mathbb{T})$  and  $\sigma, \rho \in \text{Aut}(\Sigma, [\nu])$ ,

$$(c, \sigma) \cdot (d, \rho) = (c \cdot (d \circ \sigma^{-1}), \sigma \circ \rho). \quad (2.13)$$

It follows immediately from Proposition 2.3.1 that the map

$$\begin{aligned} L^0(\Omega, \nu; \mathbb{T}) \rtimes \text{Aut}(\Sigma, [\nu]) &\longrightarrow \text{Isom}(L^p(\Omega, \nu)) \\ (c, \sigma) &\longmapsto m_c u_{p,\sigma} \end{aligned} \quad (2.14)$$

is a group homomorphism, and it is clear that it is injective. The Banach-Lamperti theorem asserts that this map is surjective when  $p \neq 2$ . That is, for  $p \neq 2$ , any surjective linear isometry can then be constructed from the two basic ones described above. The proof provided by Lamperti in [Lam58] relies on an inequality for the  $L^p$ -norms akin to the parallelogram identity on an  $L^2$ -space. We state this inequality in Lemma 2.3.2.

**Lemma 2.3.2** (Lamperti). *Let  $(\Omega, \nu)$  be a measure space, let  $1 \leq p < \infty$  and let  $\xi, \eta \in L^p(\Omega, \nu)$ .*

(i) *If  $1 \leq p \leq 2$ , we have*

$$\|\xi + \eta\|_p^p + \|\xi - \eta\|_p^p \leq 2(\|\xi\|_p^p + \|\eta\|_p^p). \quad (2.15)$$

(ii) *If  $2 \leq p < \infty$ , we have*

$$\|\xi + \eta\|_p^p + \|\xi - \eta\|_p^p \geq 2(\|\xi\|_p^p + \|\eta\|_p^p). \quad (2.16)$$

*When  $p \neq 2$ , equality holds if and only if  $\xi\eta = 0$   $\nu$ -a.e..*

*Proof.* For  $p = 2$ , the two inequalities become one equality, and we recognize this equality as the parallelogram identity. We prove below inequality (2.16) for  $p > 2$  assuming that the parallelogram identity is known. The proof of (2.15) follows the same line of reasoning, and so, we omit it.

For  $0 < \alpha < 1$  and  $x, y \geq 0$ , we have

$$(x + y)^\alpha \leq x^\alpha + y^\alpha. \quad (2.17)$$

It suffices to show the inequality in the case where  $y = 1$  and  $0 \leq x \leq 1$ . Consider the function  $f(x) = x^\alpha + 1 - (x + 1)^\alpha$  defined on  $x \in [0, 1]$ . Because  $0 < \alpha < 1$ , we see that the derivative of  $f$  is positive. Since  $f(0) = 0$ , it follows that  $f$  is non-negative on  $x \in [0, 1]$ . Inequality (2.17) follows. Observe that  $f$  is strictly positive on  $x \in (0, 1]$ , and so, (2.17) is an equality only when at least one of  $x$  and  $y$  are zero.

Now, let  $p > 2$  and let  $z, w \in \mathbb{C}$ . We apply inequality (2.17) with  $x = |z|^p$ ,  $y = |w|^p$  and  $\alpha = 2/p$  to get

$$(|z|^p + |w|^p)^{2/p} \leq |z|^2 + |w|^2. \quad (2.18)$$

Further, apply the parallelogram identity for the Hilbert space  $\mathbb{C}$  and then Hölder's inequality for the space  $\ell^{p/2}(\{0, 1\})$  to get

$$2(|z|^2 + |w|^2) = |z + w|^2 + |z - w|^2 \leq 2^{1-2/p} (|z + w|^p + |z - w|^p)^{2/p}. \quad (2.19)$$

We put together the inequalities (2.19) and (2.18) to obtain an inequality

$$2(|z|^p + |w|^p) \leq |z + w|^p + |z - w|^p, \quad (2.20)$$

which holds for all  $z, w \in \mathbb{C}$ . Clearly, (2.20) becomes an equality whenever  $zw = 0$ , that is, whenever at least one of  $z$  and  $w$  is zero. Conversely, if  $zw \neq 0$ , (2.20) is strict because (2.19) is. Now, inequality (2.16) follows from (2.20) by monotonicity of the integral and equality holds if and only if  $\xi\eta = 0$   $\nu$ -a.e..  $\square$

A consequence of the parallelogram-like inequalities of Lemma 2.3.2 is that linear isometries of  $L^p$ -spaces must preserve disjointness of functions. This fact, which we state in Lemma 2.3.3, will be of key importance in the proof of the Banach-Lamperti theorem.

**Lemma 2.3.3.** *Let  $(\Omega, \nu)$  be a measure space, let  $1 \leq p < \infty$ ,  $p \neq 2$ , and let  $T : L^p(\Omega, \nu) \rightarrow L^p(\Omega, \nu)$  be a linear isometry. Then  $T(\xi)T(\eta) = 0$  whenever  $\xi\eta = 0$  with equality  $\nu$ -almost everywhere.*

*Proof.* Suppose  $\xi, \eta \in L^p(\Omega, \nu)$  are such that  $\xi\eta = 0$ , so that the inequality of Lemma 2.3.2 becomes an equality. Then

$$\begin{aligned} \|T(\xi) + T(\eta)\|_p^p + \|T(\xi) - T(\eta)\|_p^p &= \|\xi + \eta\|_p^p + \|\xi - \eta\|_p^p \\ &= 2(\|\xi\|_p^p + \|\eta\|_p^p) \\ &= 2(\|T(\xi)\|_p^p + \|T(\eta)\|_p^p). \end{aligned}$$

As the inequality of Lemma 2.3.2 becomes an equality *only* for functions with essentially disjoint support, it follows that  $T(\xi)T(\eta) = 0$ .  $\square$

We are now ready to state and prove the Banach-Lamperti theorem. To ease the notation in the proof, we introduce the following shorthand: Given a measurable function  $f \in L^0(\Omega, \nu)$ , set

$$\{f \neq 0\} = \{\omega \in \Omega \mid f(\omega) \neq 0\}.$$

**Theorem 2.3.4** (Banach-Lamperti). *Let  $(\Omega, \Sigma, \nu)$  be a  $\sigma$ -finite measure space, let  $1 \leq p < \infty$ ,  $p \neq 2$ , and let  $T$  be a surjective isometry on  $L^p(\Omega, \nu)$ . There exist unique  $c \in L^0(\Omega, \nu; \mathbb{T})$  and  $\sigma \in \text{Aut}(\Sigma, [\nu])$  such that  $T = m_{c u_{p, \sigma}}$ .*

*Proof.* For simplicity, we prove the result for finite measure spaces. The generalization to  $\sigma$ -finite measure spaces is routine as any such space decomposes into a countable union of pairwise disjoint sets of finite measure. Assume that  $(\Omega, \Sigma, \nu)$  is a finite measure space so that  $1_A$  is integrable, for all  $A \in \Sigma$ . We define a map  $\sigma : \Sigma \rightarrow \Sigma$  as follows: For  $A \in \Sigma$ , set

$$\sigma(A) = \{T(1_A) \neq 0\}. \quad (2.21)$$

Then  $\sigma(A)$  lies in  $\Sigma$  because  $T(1_A)$  is measurable. We claim that  $\sigma : \Sigma \rightarrow \Sigma$  is an isomorphism of  $(\Sigma, [\nu])$ . To see this, observe first that, for  $A, B \in \Sigma$ , we have  $1_A = 1_B$  (with equality  $\nu$ -a.e.) if and only if  $A \sim B$ . The ‘if’-part of condition (2.2) then follows as  $T$  is a well-defined map on  $L^p(\Omega, \nu)$ . The key to show the remaining conditions is Lemma 2.3.3.

For each  $A \in \Sigma$ , consider the set  $\{T(1_\Omega)1_A \neq 0\} = \sigma(\Omega) \cap A$ . By construction, we have  $\nu(\sigma(\Omega) \cap A) = 0$  if and only if  $T(1_\Omega)1_A = 0$ . Suppose  $A \in \Sigma$  is a set such that  $\nu(\sigma(\Omega) \cap A) = 0$ . Surjectivity of  $T$  ensures the existence of a function  $\xi_A \in L^p(\Omega, \nu)$  such that  $T(\xi_A) = 1_A$ . Then Lemma 2.3.3 applied to  $T^{-1}$  implies that  $\xi_A = 1_\Omega \xi_A = 0$ . Because  $T$  is an isometry, it follows that  $\nu(A) = \|\xi_A\|_p^p = 0$ . In particular, for  $A = \sigma(\Omega)^c$ , we see that  $\nu(\sigma(\Omega)^c) = 0$ . That is,  $\sigma(\Omega) \sim \Omega$ , which is condition (2.3).

For  $A, B \in \Sigma$  essentially disjoint, i.e., with  $\nu(A \cap B) = 0$ , Lemma 2.3.3 demands that  $T(1_A)T(1_B) = 0$ , and so,  $\nu(\sigma(A) \cap \sigma(B)) = 0$ . In other words, whenever  $A, B \in \Sigma$  are essentially disjoint then so are  $\sigma(A)$  and  $\sigma(B)$ . From this, it is easy to see that  $\sigma$  preserves disjoint unions:

$$\sigma(A \cup B) = \{T(1_{A \cup B}) \neq 0\} = \{T(1_A) + T(1_B) \neq 0\} \sim \sigma(A) \cup \sigma(B). \quad (2.22)$$

Applying equation (2.22) inductively, we see that  $\sigma$  preserves unions of any finite collection of pairwise disjoint sets. Condition (2.5) follows from this using continuity of  $T$ . Further, with  $A^c$  in place of  $B$  in equation (2.22), we see that  $\nu(\sigma(A) \cap \sigma(A^c)) = 0$  and  $\sigma(\Omega) \sim \sigma(A) \cup \sigma(A^c)$ . Together with  $\sigma(\Omega) \sim \Omega$ , this implies  $\sigma(A^c) \sim \sigma(A)^c$ , which is condition (2.4).

Finally, suppose  $C \in \Sigma$  is such that  $\nu(\sigma(C)) = 0$ . Then  $T(1_C) = 0$   $\nu$ -a.e., and so,  $\|T(1_C)\|_p = 0$ . As  $T$  is an isometry, it follows that  $\nu(C) = \|1_C\|_p^p = 0$ . Now, let  $A, B \in \Sigma$  be such that  $\sigma(A) \sim \sigma(B)$ . Because  $\sigma$  preserves complements and disjoint unions, Proposition 2.2.3 gives us that  $\sigma$  preserves symmetric differences. Hence,  $\nu(\sigma(A \Delta B)) = \nu(\sigma(A) \Delta \sigma(B)) = 0$ , and it follows that  $\nu(A \Delta B) = 0$ , or, equivalently,  $A \sim B$ . This shows the ‘only if’-part of condition (2.2).

We have now shown that  $\sigma$  as defined in equation (2.21) is a homomorphism of  $(\Sigma, [\nu])$ . We proceed to reconstruct  $T$  from  $\sigma$ . Set  $h = T(1_\Omega)$ . For each  $A \in \Sigma$ , we have  $h = T(1_A) + T(1_{A^c})$  because  $T$  is linear. Since  $\sigma(A)$  and  $\sigma(A^c)$  are essentially disjoint, it follows that  $h$  and  $T(1_A)$  agree on  $\sigma(A)$ . Hence,

$$T(1_A) = T(1_A)1_{\sigma(A)} = h1_{\sigma(A)}.$$

Invoking linearity and continuity of  $T$ , and because the simple functions are dense in  $L^p(\Omega, \nu)$ , it follows that, for all  $\xi \in L^p(\Omega, \nu)$ ,

$$T(\xi) = h \cdot (\xi \circ \sigma^{-1}). \quad (2.23)$$

We now need to understand the function  $h$ . Because  $T$  is an isomorphism, we have, for  $A \in \Sigma$ ,

$$\nu(A) = \|1_A\|_p^p = \|T(1_A)\|_p^p = \int_\Omega |h1_{\sigma(A)}|^p \, d\nu = \int_{\sigma(A)} |h|^p \, d\nu.$$

At the same time, the Radon-Nikodym theorem yields

$$\nu(A) = \sigma_*\nu(\sigma(A)) = \int_{\sigma(A)} \frac{d\sigma_*\nu}{d\nu} \, d\nu.$$

Uniqueness of the Radon-Nikodym derivative then gives us that

$$|h|^p = \frac{d\sigma_*\nu}{d\nu}.$$

Set  $c = \text{sign}(h) \in L^0(\Omega, \nu; \mathbb{T})$ . We insert this into equation (2.23) to see that, for all  $\xi \in L^p(\Omega, \nu)$ ,

$$T(\xi) = c \cdot \left( \frac{d\sigma_*\nu}{d\nu} \right)^{1/p} \cdot (\xi \circ \sigma^{-1}) = m_c u_{p,\sigma}(\xi).$$

Hence,  $T = m_c u_{p,\sigma}$ . Applying the same procedure to the linear isometry  $T^{-1}$ , we find  $d \in L^0(\Omega, \nu; \mathbb{T})$  and  $\rho \in \text{Hom}(\Sigma, [\nu])$  such that  $T^{-1} = m_d u_{p,\rho}$ . Now, for  $B \in \Sigma$ ,

$$1_B = T \circ T^{-1}(1_B) = c \cdot (d \circ \sigma^{-1}) \cdot \left( \frac{d\rho_*\nu}{d\nu} \circ \sigma^{-1} \right)^{1/p} \cdot 1_{\sigma(\rho(B))}.$$

In particular, we see that  $B \sim \sigma(\rho(B))$ . Hence,  $\sigma$  satisfies condition (2.6) and is therefore not only a homomorphism but an isomorphism of  $(\Sigma, [\nu])$ .  $\square$

*Remark 2.3.5.* When  $(\Omega, \Sigma, \nu)$  in Theorem 2.3.4 is a standard Borel space and when the linear surjective isometry  $T : L^p(\Omega, \nu) \rightarrow L^p(\Omega, \nu)$  satisfies that  $\nu(\{T(1_A) \neq 0\}) = \nu(A)$ , for all  $A \in \Sigma$ , such that  $\sigma : \Sigma \rightarrow \Sigma$  defined in equation (2.21) is measure preserving, then  $\sigma$  arises from a measure preserving automorphism of  $\Omega$  as in Example 2.2.5.

## 2.4 Isometric representations on $L^p$ -spaces

With the Banach-Lamperti theorem at hand, we can now describe the isometric representation theory of a locally compact group on  $L^p$ -spaces on  $\sigma$ -finite measure spaces when  $p \neq 2$ . We construct in equation (2.24) a family of isometric representations. We then apply the Banach-Lamperti theorem in Theorem 2.4.2 to show that all isometric representations on an  $L^p$ -space on a  $\sigma$ -finite measure space must have this form when  $p \neq 2$ .

In the following, let  $G$  be a locally compact group, let  $(\Omega, \Sigma, \nu)$  be a  $\sigma$ -finite measure space, and let  $1 \leq p < \infty$  be fixed. Denote by  $\text{Hom}(G, \text{Aut}(\Sigma, [\nu]))$  the set of group homomorphisms  $G \rightarrow \text{Aut}(\Sigma, [\nu])$ . Given  $\sigma \in \text{Hom}(G, \text{Aut}(\Sigma, [\nu]))$ , we write  $\sigma_t = \sigma(t)$ , for  $t \in G$ . Further, we simplify notation and write  $t.\nu$  instead of  $(\sigma_t)_*\nu$  for the push forward measure of  $\nu$  by  $\sigma_t$ .

**Definition 2.4.1.** A  $\mathbb{T}$ -valued 1-cocycle with respect to  $\sigma \in \text{Hom}(G, \text{Aut}(\Sigma, [\nu]))$  is a map  $c : G \times \Omega \rightarrow \mathbb{T}$  such that  $c_t : \Omega \rightarrow \mathbb{T}$  is measurable, for each  $t \in G$ , and which satisfies the 1-cocycle relation, for all  $s, t \in G$ ,

$$c_{st} = c_s \cdot (c_t \circ \sigma_{s^{-1}}), \quad \nu\text{-a.e.}$$

We denote by  $Z^1(\sigma; \mathbb{T})$  the collection of all  $\mathbb{T}$ -valued 1-cocycles for  $\sigma$ .

Let  $\sigma \in \text{Hom}(G, \text{Aut}(\Sigma, [\nu]))$  and let  $c \in Z^1(\sigma; \mathbb{T})$ . Using the product defined in equation (2.13), we get, for each pair  $s, t \in G$ ,

$$(c_{st}, \sigma_{st}) = (c_s \cdot (c_t \circ \sigma_s^{-1}), \sigma_s \circ \sigma_t) = (c_s, \sigma_s) \cdot (c_t, \sigma_t).$$

Hence,  $t \mapsto (c_t, \sigma_t)$  defines a group homomorphism  $G \rightarrow L^0(\Omega, \nu; \mathbb{T}) \rtimes \text{Aut}(\Sigma, [\nu])$ . We associate to the pair  $\sigma \in \text{Hom}(G, \text{Aut}(\Sigma, [\nu]))$  and  $c \in Z^1(\sigma, \mathbb{T})$  a group homomorphism  $\pi_{p, \sigma, c} : G \rightarrow \text{Isom}(L^p(\Omega, \nu))$  by composing the map  $t \mapsto (c_t, \sigma_t)$  with the group homomorphism of equation (2.14) from Section 2.3:

$$\begin{array}{ccc} G & \xrightarrow{\pi_{p, \sigma, c}} & \text{Isom}(L^p(\Omega, \nu)) \\ & \searrow^{t \mapsto (c_t, \sigma_t)} & \nearrow_{(c, \sigma) \mapsto m_{c, \nu, \sigma}} \\ & & L^0(\Omega, \nu; \mathbb{T}) \rtimes \text{Aut}(\Sigma, [\nu]) \end{array}$$

Explicitly, for  $\xi \in L^p(\Omega, \nu)$ ,  $t \in G$  and  $\omega \in \Omega$ ,

$$\pi_{p,\sigma,c}(t)\xi(\omega) = c_t(\omega) \left( \frac{d t.\nu}{d\nu}(\omega) \right)^{1/p} \xi \circ \sigma_t^{-1}(\omega). \quad (2.24)$$

Then  $\pi_{p,\sigma,c} : G \rightarrow \text{Isom}(L^p(\Omega, \nu))$  is an isometric representation whenever it is strongly continuous. If  $G$  is discrete, strong continuity is automatic.

**Theorem 2.4.2.** *Let  $\pi$  be an isometric representation of a locally compact group  $G$  on  $L^p(\Omega, \nu)$ , with  $(\Omega, \Sigma, \nu)$  a  $\sigma$ -finite measure space and  $1 \leq p < \infty$ ,  $p \neq 2$ . There exist unique  $\sigma \in \text{Hom}(G, \text{Aut}(\Sigma, [\nu]))$  and  $c \in Z^1(\sigma; \mathbb{T})$  such that  $\pi = \pi_{p,\sigma,c}$ .*

*Proof.* For each group element  $t \in G$ , the Banach-Lamperti theorem yields unique  $c_t \in L^0(\Omega, \nu; \mathbb{T})$  and  $\sigma_t \in \text{Aut}(\Sigma, [\nu])$  such that  $\pi(t) = m_{c_t} u_{p,\sigma_t}$ . We obtain a map  $\sigma : G \rightarrow \text{Aut}(\Sigma, [\nu])$  given by  $\sigma(t) = \sigma_t$ , for  $t \in G$ , and a map  $c : G \times \Omega \rightarrow \mathbb{T}$  given by  $c(t, \omega) = c_t(\omega)$ , for  $t \in G$  and  $\omega \in \Omega$ . Since  $\pi$  is a group homomorphism, we have, for each pair  $s, t \in G$ ,

$$m_{c_{st}} u_{p,\sigma_{st}} = \pi(st) = \pi(s)\pi(t) = m_{c_s} u_{p,\sigma_s} m_{c_t} u_{p,\sigma_t} = m_{c_s \cdot (c_t \circ \sigma_s^{-1})} u_{p,\sigma_s \circ \sigma_t}.$$

The uniqueness part of the Banach-Lamperti theorem enforces that  $\sigma_{st} = \sigma_s \circ \sigma_t$  and  $c_{st} = c_s \cdot (c_t \circ \sigma_s^{-1})$ , for all  $s, t \in G$ . That is,  $\sigma : G \rightarrow \text{Aut}(\Sigma, [\nu])$  is a group homomorphism,  $c : G \times \Omega \rightarrow \mathbb{T}$  is a  $\mathbb{T}$ -valued 1-cocycle for  $\sigma$ , and  $\pi = \pi_{p,\sigma,c}$ .  $\square$

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# Chapter 3

## Analytic properties of locally compact groups

We recall the definitions of the analytic properties studied in the papers constituting this thesis. Amenability, which is the topic of paper IV, is treated in section 3.1. Property  $(T_{\mathcal{E}})$ , with  $\mathcal{E}$  a class of Banach spaces, is treated in section 3.2 and is the topic of paper II and III.

### 3.1 Amenability

We say that the locally compact group  $G$  admits an invariant mean if there exists a positive linear functional on  $L^\infty(G)$  normalized on the identity and which is invariant for the left translation action of  $G$  on  $L^\infty(G)$ .

**Definition 3.1.1.** A locally compact group is called *amenable* if it admits an invariant mean.

In this section, we review a selection of its many equivalent characterizations, focusing on those connecting with the unitary representation theory. We refer to [Run02], [Pie84], Section 2.6 in [BO08] or Appendix G in [BLV08] for thorough introductions to the topic of amenability and its many connections.

Given unitary representations  $\pi$  and  $\rho$  of  $G$ , recall from section 1.2 that  $\pi$  is said to be *weakly contained* in  $\rho$ , in symbols  $\pi \prec \rho$ , if  $\|\pi(f)\| \leq \|\rho(f)\|$ , for all  $f \in L^1(G)$ . Deep results of Hulanicki in [Hul66] and Reiter in [Rei65] yield a representation theoretic characterization of amenability in terms of weak containment. Denote by  $1_G$  the trivial representation of  $G$  on  $\mathbb{C}$  and by  $\lambda$  the left-regular representation of  $G$  on  $L^2(G)$  (see Example 1.1.4).

**Theorem 3.1.2** (Hulanicki-Reiter). *A locally compact group  $G$  is amenable if and only if  $1_G \prec \lambda$ .*

The work of Hulanicki and Reiter form the basis of our representation theoretic understanding of amenability, and, by extension, our  $C^*$ -algebraic understanding of it. We exhibit in Theorem 3.1.4 three well-known  $C^*$ -algebraic characterizations

of amenability. Among the abundance of characterizations of amenability, we have picked out these three specifically to create a theorem that will give context to Theorem B of paper IV. The fact that the  $C^*$ -algebraic properties of Theorem 3.1.4 are, indeed, equivalent to the property of being amenable follows without too much effort from Theorem 3.1.2. We shall need the following lemma, which is Corollary F.3.3 in [BLV08]:

**Lemma 3.1.3.** *Let  $G$  be a locally compact group. Then  $1_G \prec \lambda$  if and only if  $\pi \prec \lambda$ , for every unitary representation  $\pi$  of  $G$ .*

**Theorem 3.1.4.** *Let  $G$  be a locally compact group. The following are equivalent:*

- (i)  $G$  is amenable.
- (ii)  $C^*(G) = C_r^*(G)$ .
- (iii)  $C_r^*(G)' \cong B(G)$  canonically and isometrically.
- (iv)  $1_G$  extends to a  $*$ -representation of  $C_r^*(G)$ .

*Proof.* (i) $\Rightarrow$ (ii): If  $G$  is amenable, then  $\pi \prec \lambda$ , for every unitary representation  $\pi$  of  $G$ , by Theorem 3.1.2 and Lemma 3.1.3. Then  $\|\pi(f)\| \leq \|\lambda(f)\|$ , for every unitary representation  $\pi$  of  $G$  and every  $f \in L^1(G)$ . Since the universal  $C^*$ -norm is the supremum over  $C^*$ -norms arising from unitary representations, it follows that the universal  $C^*$ -norm is dominated by the reduced. Hence, the universal and the reduced  $C^*$ -norms must agree.

(ii) $\Rightarrow$ (iii): Recall that the Banach space dual of the universal group  $C^*$ -algebra of  $G$  is canonically and isometrically isomorphic to the Fourier-Stieltjes algebra,  $B(G)$ . Hence, (iii) is the dual picture of (ii).

(iii) $\Rightarrow$ (iv): The constant function 1 on  $G$  always belongs to  $B(G)$ ; it corresponds to  $1_G$  viewed as a linear functional on  $C^*(G)$  via the canonical isomorphism between  $B(G)$  and  $C^*(G)'$ . Hence,  $1_G$  extends to a linear functional on  $C_r^*(G)$ . Since  $1_G$  is a  $*$ -representation of  $L^1(G)$ , which is dense in  $C_r^*(G)$ , its extension to  $C_r^*(G)$  will still be a  $*$ -representation.

(iv) $\Rightarrow$ (i): If  $1_G$  extends to a  $*$ -representation of  $C_r^*(G)$  then  $1_G$  is weakly contained in the left regular representation by Proposition 1.2.5. It follows from Theorem 3.1.2 that  $G$  is amenable.  $\square$

## 3.2 Kazhdan-type rigidity

We give here an introduction to Kazhdan-type rigidity for locally compact groups acting by isometries on Banach spaces from a specified class. We take a different but equivalent approach compared to that of Bader, Furman, Gelander and Monod in [BFGM07]. The classical property (T) of Kazhdan arises as a special case.

Throughout this section, we let  $G$  be a locally compact group.

**Definition 3.2.1.** Let  $(\pi, E)$  be an isometric representation of  $G$  on a Banach space  $E$ .

- (i) A vector  $\xi \in E$  is said to be *invariant* if, for all  $t \in G$ ,

$$\pi(t)\xi = \xi.$$

The collection of all such vectors forms a closed subspace of  $E$ , which we denote  $E^\pi$ .

- (ii) A net of vectors  $(\xi_i)_{i \in I}$  in  $E$  is said to be *almost invariant* if, for every compact subset  $K \subset G$ ,

$$\sup_{t \in K} \|\pi(t)\xi_i - \xi_i\|_E \rightarrow 0.$$

Recall that the distance from a vector  $\xi \in E$  to a subset  $S \subset E$  is given by  $\text{dist}(\xi, S) = \inf_{\eta \in S} \|\xi - \eta\|_E$ . Given an isometric representation  $(\pi, E)$  of  $G$  on a Banach space  $E$  and a vector  $\xi \in E$ , a standard application of the triangle inequality yields, for each  $\eta \in E^\pi$ , the upper bound  $\|\pi(t)\xi - \xi\| \leq 2\|\xi - \eta\|$ . Take the infimum of the right hand side over  $E^\pi$ , and we see that

$$\|\pi(t)\xi - \xi\| \leq 2 \inf_{\eta \in E^\pi} \|\xi - \eta\| = 2 \text{dist}(\xi, E^\pi).$$

Hence, for a net  $(\xi_i)_{i \in I}$  of unit vectors in  $E$ , if  $(\xi_i)_{i \in I}$  approaches  $E^\pi$  in the sense that the distance  $\text{dist}(\xi_i, E^\pi)$  converges to zero, or, equivalently, when there exists a net  $(\eta_i)_{i \in I}$  in  $E^\pi$  such that  $\|\xi_i - \eta_i\|$  converges to zero, then  $(\xi_i)_{i \in I}$  must be a net of almost invariant vectors. The converse need not be true. That is, nets of almost invariant vectors do not a priori need to approach  $E^\pi$ . When the converse implication does hold, we say that  $G$  obeys a Kazhdan type rigidity property with respect to the representation  $(\pi, E)$ . Given a class of Banach spaces  $\mathcal{E}$ , we define property  $(T_{\mathcal{E}})$  (read: ‘property (T) with respect to the class  $\mathcal{E}$ ’) in Definition 3.2.2 below. Note that the formulation given here is not the original one by Bader, Furman, Gelander and Monod from [BFGM07]. We shall show in Theorem 3.2.6 later in this section that the two definitions of property  $(T_{\mathcal{E}})$  are equivalent.

**Definition 3.2.2** (Property  $(T_{\mathcal{E}})$ ). Let  $\mathcal{E}$  be a class of Banach spaces. A locally compact group  $G$  has *property  $(T_{\mathcal{E}})$*  if, whenever an isometric representation  $(\pi, E)$  of  $G$  with  $E$  in the class  $\mathcal{E}$  admits a net of almost invariant unit vectors  $(\xi_i)_{i \in I}$ , there exists a net of invariant vectors  $(\eta_i)_{i \in I}$  such that  $\|\xi_i - \eta_i\|_E \rightarrow 0$ .

*Remark 3.2.3.* With  $\mathcal{H}$  denoting the class of complex Hilbert spaces, property  $(T_{\mathcal{H}})$  recovers Kazhdan’s property (T). See, e.g., Proposition 1.1.9 in [BLV08].

Definition 3.2.2 above is a qualitative formulation of property  $(T_{\mathcal{E}})$ . Indeed, it does not quantify how close a vector is to the subspace of invariant vectors given that it is in an appropriate sense close to being invariant. A quantitative formulation of property  $(T_{\mathcal{E}})$  also appears in the literature, e.g., Ng’s generalization of property (T) to  $C^*$ -algebras in [Ng13] is quantitative. We shall briefly discuss this quantitative approach here.

Let  $(\pi, E)$  be an isometric representation of the locally compact group  $G$ . A pair  $(Q, \delta)$  with  $Q \subset G$  and  $\delta > 0$  is called an  $\varepsilon$ -Kazhdan pair for  $\pi$  with  $\varepsilon > 0$  if, whenever  $\xi \in E$  with  $\|\xi\| = 1$  satisfies  $\sup_{t \in Q} \|\pi(t)\xi - \xi\| < \delta$ , then  $\inf_{\eta \in E^\pi} \|\xi - \eta\| < \varepsilon$ . We say that an  $\varepsilon$ -Kazhdan pair  $(Q, \delta)$  is compact if  $Q$  is a compact subset. We expect that the following result is well-known to experts. We provide details of the proof for completeness.

**Theorem 3.2.4.** *Let  $G$  be a locally compact group and let  $\mathcal{E}$  be a class of Banach spaces. Then  $G$  has property  $(T_{\mathcal{E}})$  if and only if, for every isometric representation  $(\pi, E)$  of  $G$  with  $E \in \mathcal{E}$  and for every  $\varepsilon > 0$ , there exists a compact  $\varepsilon$ -Kazhdan pair for  $\pi$ .*

*Proof.* Suppose that  $G$  does not have property  $(T_{\mathcal{E}})$ . Then we can find an isometric representation  $(\pi, E)$  of  $G$  and a net of almost invariant unit vectors  $(\xi_i)_{i \in I}$  in  $E$  such that  $\inf_{\eta \in E^\pi} \|\xi_i - \eta\| \geq \varepsilon$ , for all  $i \in I$  and for some  $\varepsilon > 0$  not depending on  $i$ . Since  $(\xi_i)_{i \in I}$  is almost invariant, there is, for all  $Q \subset G$  compact and  $\delta > 0$ , an index  $i_0 \in I$  such that  $\sup_{t \in Q} \|\pi(t)\xi_{i_0} - \xi_{i_0}\| < \delta$ . It follows that there are no compact  $\varepsilon$ -Kazhdan pairs for  $\pi$ . This proves the ‘if’-part of the theorem. For the ‘only if’-part of the theorem, suppose there exists an  $\varepsilon > 0$  such that, for all compact subsets  $Q \subset G$  and all  $\delta > 0$ , one can find a unit vector  $\xi \in E$  for which  $\sup_{t \in Q} \|\pi(t)\xi - \xi\| < \delta$  but  $\inf_{\eta \in E^\pi} \|\xi - \eta\| \geq \varepsilon$ . Let  $I$  be the collection of all tuples  $(Q, \delta)$  with  $Q \subset G$  compact and  $\delta > 0$  directed with the relation  $(Q_1, \delta_1) \preceq (Q_2, \delta_2)$  if  $Q_1 \subset Q_2$  and  $\delta_1 \geq \delta_2$ . Then, for each  $(Q, \delta) \in I$ , pick a unit vector  $\xi_{(Q, \delta)} \in E$  such that  $\sup_{t \in Q} \|\pi(t)\xi - \xi\| < \delta$  and  $\inf_{\eta \in E^\pi} \|\xi_i - \eta\| \geq \varepsilon$ . Then  $(\xi_i)_{i \in I}$  is almost invariant but cannot approach  $E^\pi$ . Hence, under these assumptions  $G$  does not have property  $(T_{\mathcal{E}})$ .  $\square$

Property  $(T_{\mathcal{E}})$ , as defined by Bader, Furman, Gellander and Monod in [BFGM07], is the property that the relevant quotients  $E/E^\pi$  lack nets of almost invariant unit vectors. We show in Theorem 3.2.6 that their definition is equivalent to the one we give in Definition 3.2.2. The proof is based on Lemma 18 in [Tan17] by Tanaka, which we state and reprove in Lemma 3.2.5. Tanaka states the lemma with the extra assumption that the group is second countable. This assumption, however, can be removed.

Let  $(\pi, E)$  be an isometric representation of  $G$ . For an element  $\xi \in E$ , we denote by  $[\xi]$  its equivalence class in the quotient space  $E/E^\pi$ . Its norm is given by  $\inf_{\eta \in E^\pi} \|\xi - \eta\|$ . The *quotient representation* on  $E/E^\pi$  is the isometric representation  $\tilde{\pi}$  given by  $\tilde{\pi}(t)[\xi] = [\pi(t)\xi]$ , for  $t \in G$  and  $\xi \in E$ .

**Lemma 3.2.5** (Tanaka). *Let  $G$  be a locally compact group, let  $(\pi, E)$  be an isometric representation of  $G$  on a Banach space  $E$ , and denote by  $\tilde{\pi}$  the quotient representation on  $E/E^\pi$ . For every  $t \in G$  and  $\xi \in E$ , it holds that*

$$\|\tilde{\pi}(t)[\xi] - [\xi]\|_{E/E^\pi} \leq \|\pi(t)\xi - \xi\|_E \leq 2\|\tilde{\pi}(t)[\xi] - [\xi]\|_{E/E^\pi}.$$

*Proof.* The first inequality follows trivially from the construction of the quotient norm. For the second inequality, let  $\xi \in E$  and  $t \in G$ . Observe that, for every

$\eta \in E^\pi$  and every  $n \in \mathbb{N}$ ,

$$\begin{aligned} \|\eta\|_E &\leq \left\| \frac{\pi(t^n)\xi - \xi}{n} + \eta \right\|_E + \left\| \frac{\pi(t^n)\xi - \xi}{n} \right\|_E \\ &\leq \frac{1}{n} \sum_{k=1}^n \left\| \pi(t^k)\xi - \pi(t^{k-1})\xi + \eta \right\|_E + \frac{\|\pi(t^n)\xi - \xi\|_E}{n} \\ &= \frac{1}{n} \sum_{k=1}^n \left\| \pi(t^{k-1})(\pi(t)\xi - \xi + \eta) \right\|_E + \frac{2\|\xi\|_E}{n} \\ &= \|\pi(t)\xi - \xi + \eta\|_E + \frac{2\|\xi\|_E}{n}. \end{aligned}$$

Letting  $n$  approach infinity, it follows that, for every  $\eta \in E^\pi$ ,

$$\|\pi(t)\xi - \xi\|_E \leq \|\pi(t)\xi - \xi + \eta\|_E + \|\eta\|_E \leq 2\|\pi(t)\xi - \xi + \eta\|_E.$$

When taking the infimum over  $\eta \in E^\pi$ , the second inequality of the lemma follows.  $\square$

We are now ready to show that Definition 3.2.2 is equivalent to the definition of property  $(T_\mathcal{E})$  by Bader, Furman, Gelander and Monod, as promised.

**Theorem 3.2.6.** *Let  $G$  be a locally compact group and let  $\mathcal{E}$  be a class of Banach spaces. Then  $G$  has property  $(T_\mathcal{E})$  if and only if, for every isometric representation  $(\pi, E)$  of  $G$  with  $E$  in the class  $\mathcal{E}$ , the quotient  $E/E^\pi$  admits no nets of almost invariant unit vectors.*

*Proof.* Assume  $G$  does not have property  $(T_\mathcal{E})$ . We can then find an isometric representation  $(\pi, E)$  of  $G$  with  $E \in \mathcal{E}$  and a net of almost invariant unit vectors  $(\xi_i)_i$  in  $E$  which does not approach  $E^\pi$ . Take  $\delta > 0$  such that, for all indices  $i$ ,

$$\|[\xi_i]\|_{E/E^\pi} = \inf_{\eta \in E^\pi} \|\xi_i - \eta\|_E > \delta.$$

Set  $\xi'_i = \xi_i / \|\xi_i\|_E$ . Then, by the left-most inequality of Lemma 3.2.5, we have, for each  $t \in G$ ,

$$\|\tilde{\pi}(t)[\xi'_i] - [\xi'_i]\|_{E/E^\pi} = \frac{\|\tilde{\pi}(t)[\xi_i] - [\xi_i]\|_{E/E^\pi}}{\|\xi_i\|_E} \leq \frac{1}{\delta} \|\pi(t)\xi_i - \xi_i\|_E.$$

Hence,  $([\xi'_i])_i$  is a net of almost invariant unit vectors in  $E/E^\pi$ . This shows the ‘if’-part of the theorem. For the converse, suppose  $([\xi_i])_i$  is a net of almost invariant unit vectors in  $E/E^\pi$ . Given  $\delta > 0$ , we can pick the representative  $\xi_i$  such that  $1 \leq \|\xi_i\|_E \leq 1 + \delta$ . Set  $\xi'_i = \xi_i / \|\xi_i\|_E$  so that  $(\xi'_i)_i$  is a net of unit vectors in  $E$ . Then, for each  $\eta \in E^\pi$ ,

$$\|\xi'_i - \eta\|_E = \frac{1}{\|\xi_i\|_E} \|\xi_i - \|\xi_i\|_E \eta\|_E \geq \frac{1}{\|\xi_i\|_E} \|[\xi_i]\|_{E/E^\pi} \geq \frac{1}{1 + \delta}.$$

It follows that  $(\xi'_i)_i$  does not approach  $E^\pi$ . Further, by the right-most inequality of Lemma 3.2.5, we have, for each  $t \in G$ ,

$$\|\pi(t)\xi'_i - \xi'_i\|_E \leq \|\pi(t)\xi_i - \xi_i\|_E \leq 2\|\tilde{\pi}(t)[\xi_i] - [\xi_i]\|_{E/E^\pi}.$$

Thus,  $(\xi'_i)_i$  is almost invariant. We conclude that  $G$  does not have property  $(T_\mathcal{E})$ . This shows the ‘only if’-part of the theorem.  $\square$

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# Chapter 4

## Paper summaries

### Paper I

In paper I, titled *Unitary  $L^{p+}$ -representations of almost automorphism groups*, we study integrability properties of unitary representations induced from an open subgroup. A unitary representation is said to be an  $L^p$ -representation if sufficiently many of its matrix coefficients are  $L^p$ -integrable, and it is an  $L^{p+}$ -representation if it is an  $L^{p+\varepsilon}$ -representation, for all  $\varepsilon > 0$ . Both of these properties are preserved under the process of inducing representations from open subgroups. However, it could happen that the induced representation has better integrability properties, i.e., a representation induced from an  $L^p$ -representation could also be an  $L^q$ -representation, for some  $q < p$ , even if the original representation is not. There is no known example where this happens when inducing from an open subgroup, but it does happen for example when inducing  $L^p$ -representations of  $\mathbb{F}_2$  to higher rank Lie groups for sufficiently large values of  $p$ . Overall, the connection between the optimal  $L^p$ - and  $L^{p+}$ -integrability of induced representations and of the representations they are induced from is not well understood. The main result of paper I is that, when inducing from an open subgroup with the Kunze-Stein property, the  $L^{p+}$ -integrability cannot improve. As an application of this, we obtain an insight into the optimal  $L^{p+}$ -integrability properties of a specific class of representations of the (colored) Neretin groups, or, more generally, of any  $H$ -honest almost automorphism group of a tree  $T_d$  of degree  $d \geq 3$ , where  $H$  is a non-compact closed subgroup of  $\text{Aut}(T_d)$  acting transitively on the boundary of  $T_d$  and satisfying Tits' independence property. In particular, for any such group and any exponent  $2 < q < p < \infty$ , there is an  $L^{p+}$ -representation which is not an  $L^q$ -representation. We remark that the final conclusion also follows via  $C^*$ -algebraic methods by observing that the  $L^{p+}$ - $C^*$ -algebras are canonically non-isomorphic. However, the  $C^*$ -algebraic argument does not offer any insight into the optimal  $L^{p+}$ -integrability of specific representations.

## Paper II

The purpose of paper II, titled *Property (T) for Banach algebras* is to define and study analogues of Kazhdan's property (T) for Banach algebras acting on Banach spaces. Building on the work of Bekka and Ng in [Bek05], [Ng13] and [BN19], we define two versions of property (T) for Banach algebras: *property*  $(T_{\mathcal{E}})$  and *weak property*  $(T_{\mathcal{E}})$ , where  $\mathcal{E}$  denotes a class of Banach spaces. When the algebra in question is a  $C^*$ -algebra and when  $\mathcal{E}$  is the class of Hilbert spaces, we recover the two versions of property (T) for  $C^*$ -algebras introduced by Bekka and Ng. For von Neumann algebras, the weaker version recovers property (T) in the sense of Connes (see [Con80]). After defining property  $(T_{\mathcal{E}})$  and its weaker relative for Banach algebras, it is natural to ask whether property  $(T_{\mathcal{E}})$  for a locally compact group  $G$  as defined by Bader, Furman, Gelander and Monod in [BFGM07] is captured by our property  $(T_{\mathcal{E}})$  for a Banach algebra constructed from  $G$  and similarly for weak property  $(T_{\mathcal{E}})$ . Our first main result of the paper is to answer this question affirmatively. For the class of  $L^p$ -spaces, it is desirable to consider the  $p$ -pseudofunction algebra or its symmetrized version rather than the pseudofunction algebra associated to all isometric  $L^p$ -representations because the former is a more tractable object. Our second main result establishes that a discrete group has property  $(T_{L^p})$  if and only if its (symmetrized)  $p$ -pseudofunction algebra does. This is a partial generalization of a result by Bekka and Ng in [BN19] who show the analogous result in the Hilbert space setting for the class of IN groups. For non-discrete IN groups and  $p \neq 2$ , we get the slightly weaker results that property  $(T_{L^p})$  for the (symmetrized)  $p$ -pseudofunction algebra implies weak property  $(T_{L^p})$  for the group. Finally, using tools from ergodic theory, we explore some connections between property  $(T_{L^p})$  and weak property  $(T_{SL^p})$  for a discrete group and its (symmetrized)  $p$ -pseudofunction algebra. We show that weak property  $(T_{SL^p})$  implies property  $(T_{L^p})$  for a discrete group, and that weak property  $(T_{SL^p})$  for its (symmetrized)  $p$ -pseudofunction algebra is intermediate to the two.

## Paper III

Paper III, titled *Weak property  $(T_{L^p})$  for discrete groups*, concerns the definition of property  $(T_{L^p})$  for discrete groups, where  $1 \leq p < \infty$  and  $L^p$  denotes the class of  $L^p$ -spaces on  $\sigma$ -finite measure spaces. Based on an often used formulation of Kazhdan's property (T), we define the following Kazhdan-type rigidity property with respect to a given class of Banach spaces  $\mathcal{E}$ : A locally compact group  $G$  has weak property  $(T_{\mathcal{E}})$  if, whenever a space in  $\mathcal{E}$  admits a net of almost  $G$ -invariant unit vectors, there must be a non-zero  $G$ -invariant vector. As the name suggests, this is an a priori weaker notion than property  $(T_{\mathcal{E}})$  as it is defined by Bader, Furman, Gelander and Monod in [BFGM07]. For the class  $\mathcal{H}$  of Hilbert spaces, it is well-known that weak property  $(T_{\mathcal{H}})$  coincides with Kazhdan's property (T). In general, whether weak property  $(T_{\mathcal{E}})$  implies property  $(T_{\mathcal{E}})$  may depend on the class  $\mathcal{E}$ . Our main aim in this paper is to show that, for discrete groups, weak

$(T_{L^p})$  is equivalent to property  $(T_{L^p})$ .

In the first part of the paper, we study the subgroup of 1-coboundaries for an ergodic p.m.p. action. Via an application of the Open Mapping Theorem for Polish groups, we establish a connection between the closure of this subgroup and strong ergodicity of the action. Together with the Connes-Weiss characterization of property  $(T)$ , it follows that if the subgroup of 1-coboundaries is closed for all ergodic p.m.p. actions then the group in question must have property  $(T)$ .

Next, we analyze when a representation associated to a p.m.p. action and a 1-cocycle admits a non-zero invariant vector, respectively, a net of almost invariant unit vectors. We show that the former happens exactly when the 1-cocycle is a 1-coboundary, and the latter when it is a limit of 1-coboundaries. With these insights at hand, we can apply our result from the first part of the paper to show the main result: the equivalence of weak property  $(T_{L^p})$  and property  $(T_{L^p})$ .

Finally, we observe that, if for any isometric representation  $(\pi, L^p(\Omega, \nu))$ , the quotient  $L^p(\Omega, \nu)/L^p(\Omega, \nu)^\pi$  would be equivariantly and boundedly isomorphic to an  $L^p$ -space on a  $\sigma$ -finite measure space, this would allow for a much shorter proof of the equivalence of weak property  $(T_{L^p})$  and property  $(T_{L^p})$  compared to the one we give in the paper. In the final part of the paper, we show that when  $\pi$  comes from an ergodic p.m.p. action on a diffuse standard probability space, the quotient  $L^p(\Omega, \nu)/L^p(\Omega, \nu)^\pi$  is for most values of  $p$  not isometrically isomorphic to an  $L^p$ -space on a  $\sigma$ -finite measure space. This partially refutes this alternative proof strategy.

## Paper IV

In paper IV, titled *Symmetrized pseudofunction algebras from  $L^p$ -representations and amenability of locally compact groups*, we collect and review characterizations of amenability of locally compact groups in terms of the  $p$ -pseudofunction algebra of Herz and its dual, and we establish analogue results for the symmetrized  $p$ -pseudofunction algebra.

The starting point of the paper is the diagram below consisting of canonical contractive  $*$ -homomorphisms and for  $G$  a locally compact group. The existence of the right edge is shown by Samei and Wiersma in [SW24]. In the first part of this paper, we establish the left edge via an application of tools from complex interpolation theory.

$$\begin{array}{ccc}
 & L^1(G) & \\
 & \swarrow & \searrow \\
 F_{L^p}^*(G) & \xrightarrow{\quad} & F_{\lambda_p}^*(G) \\
 \swarrow & & \searrow \\
 C^*(G) & \xrightarrow{\quad} & C_r^*(G)
 \end{array}$$

With this diagram in mind, we think of  $F_{L^p}^*(G)$  as interpolating between  $L^1(G)$  and  $C^*(G)$  with the parameter  $1 \leq p \leq 2$ . Recall from Theorem 3.1.4 that a

locally compact group  $G$  is amenable if and only if its full and its reduced group  $C^*$ -algebras coincide. The above diagram therefore motivates us to ask for a similar characterization of amenability with  $F_{L^p}^*(G)$  in place of  $C^*(G)$  and  $F_{\lambda_p}^*(G)$  in place of  $C_r^*(G)$ : If  $G$  is amenable, must  $F_{L^p}^*(G)$  coincide with  $F_{\lambda_p}^*(G)$ ? And, conversely, in order for  $G$  to be amenable, does it suffice that  $F_{L^p}^*(G)$  and  $F_{\lambda_p}^*(G)$  coincide for some  $p > 1$ ? These questions are further motivated by a result by Samei and Wiersma in [SW24], who show that  $G$  is amenable if and only if the trivial representation extends to a  $*$ -representation of  $F_{\lambda_p}^*(G)$ , for some  $p > 1$ . Furthermore, it is known from work of Gardella and Thiel in [GT14] that the analogous questions in the non-symmetrized setting have affirmative answers. We show in final part of the paper that the answer to both of these questions is also affirmative in the symmetrized setting. The path to our characterization of amenability in terms of  $F_{\lambda_p}^*(G)$  leads us to study its Banach space dual, which we do in the middle part of the paper.

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# Papers



# Paper I

## Unitary $L^{p+}$ -representations of almost automorphism groups

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Published in *Comptes Rendus Mathématique* 362 (2024), pp. 245–249.

**Abstract** Let  $G$  be a locally compact group with an open subgroup  $H$  with the Kunze–Stein property, and let  $\pi$  be a unitary representation of  $H$ . We show that the representation  $\tilde{\pi}$  of  $G$  induced from  $\pi$  is an  $L^{p+}$ -representation if and only if  $\pi$  is an  $L^{p+}$ -representation. We deduce the following consequence for a large natural class of almost automorphism groups  $G$  of trees: For every  $p \in (2, \infty)$ , the group  $G$  has a unitary  $L^{p+}$ -representation that is not an  $L^{q+}$ -representation for any  $q < p$ . This in particular applies to the Neretin groups.

### I.1 Induction of unitary $L^{p+}$ -representations

Let  $G$  be a locally compact group with a closed subgroup  $H$ . Given a unitary representation  $\pi$  of  $H$ , it can be induced to  $G$  in order to get a unitary representation of  $G$ . Technically, this procedure is much easier if  $H$  is not only closed, but also open in  $G$ . In that case, it is well known that the restriction  $\tilde{\pi}|_H$  (restricted to  $H$  as a function) contains the representation  $\pi$ .

We briefly recall the construction of induced representations, and we refer to [KT13; BH20] for further details. Let  $H$  be an open subgroup of  $G$ , and let  $\{t_i\}_{i \in I}$  be a maximal set in  $G$  consisting of left coset representatives. Let  $(\pi, V)$  be a unitary representation of  $H$ . Let

$$\tilde{V} = \{f: G \rightarrow V \mid f(gh) = \pi(h^{-1})f(g) \forall g \in G \forall h \in H \text{ and } \sum_{i \in I} \|f(t_i)\|^2 < \infty\}.$$

With respect to the inner product  $\langle f, f' \rangle = \sum_{i \in I} \langle f(t_i), f'(t_i) \rangle$ , this is a Hilbert space. The induced representation  $(\tilde{\pi}, \tilde{V})$  is defined by  $(\tilde{\pi}(g)f)(g') = f(g^{-1}g')$ .

Given a unitary representation  $(\pi, V)$  of  $G$ , a function of the form  $\pi_{v,w}: g \mapsto \langle \pi(g)v, w \rangle$ , where  $v, w \in V$ , is called a matrix coefficient of  $\pi$ . This short note is concerned with the behaviour of integrability properties of such matrix coefficients under the procedure of induction of representations from open subgroups.

**Definition I.1.1.** Let  $p \in [1, \infty)$ . A unitary representation  $(\pi, V)$  of a locally compact group  $G$  is an  $L^p$ -representation if there is a dense linear subspace  $V_0 \subset V$  such that for all  $v, w \in V_0$ , we have  $\pi_{v,w} \in L^p(G)$ . The representation  $\pi$  is an  $L^{p+}$ -representation if it is an  $L^{p+\varepsilon}$ -representation for all  $\varepsilon > 0$ .

The property of being an  $L^p$ -representation (or  $L^{p+}$ -representation) is preserved under induction of representations. We briefly recall the proof, which is based on Theorem 2.4 in [Wie16], in which Wiersma shows this fact in the setting of discrete groups. The same proof works in our setting and we claim no originality here.

**Proposition I.1.2.** Let  $G$  be a locally compact group,  $H$  an open subgroup of  $G$ , and  $p \in [1, \infty)$ . Let  $(\pi, V)$  be a unitary representation of  $H$  and  $(\tilde{\pi}, \tilde{V})$  the associated induced representation. If  $\pi$  is an  $L^p$ -representation (resp.  $L^{p+}$ -representation), then  $\tilde{\pi}$  is an  $L^p$ -representation (resp.  $L^{p+}$ -representation).

*Proof.* Let  $V_0$  be a dense linear subspace of  $V$  such that for all  $v, w \in V_0$ , the matrix coefficient  $\pi_{v,w}$  lies in  $L^p(H)$ . Let  $\{t_i\}_{i \in I}$  be as above, and let  $\tilde{V}_0$  be the subspace of  $\tilde{V}$  consisting of functions  $f$  for which  $f(t_i) \in V_0$  for all  $i \in I$  and such that  $f(t_i)$  is non-zero for only finitely many  $i \in I$ . Then  $\tilde{V}_0$  is a dense linear subspace of  $\tilde{V}$ .

For each  $i \in I$ , denote by  $\tilde{V}_0^i$  the linear subspace of  $\tilde{V}$  consisting of all functions  $f$  with support in  $t_i H$  and such that  $f(t_i) \in V_0$ . Then  $\tilde{V}_0$  consists of finite linear combinations of functions from (different) spaces  $\tilde{V}_0^i$ . For  $i, j \in I$  and  $f \in \tilde{V}_0^i$  and  $f' \in \tilde{V}_0^j$ , consider the matrix coefficient  $\tilde{\pi}_{f,f'}$ . Set  $w = f(t_i) \in V_0$  and  $w' = f'(t_j) \in V_0$ , and denote the modular function on  $G$  (and its restriction to  $H$ ) by  $\Delta$ . Then

$$\begin{aligned} \|\tilde{\pi}_{f,f'}\|_p^p &= \int_G \left| \langle \tilde{\pi}(g)f, f' \rangle \right|^p d\mu_G(g) = \int_G \left| \sum_{k \in I} \langle f(g^{-1}t_k), f'(t_k) \rangle \right|^p d\mu_G(g) \\ &= \int_G \left| \langle f(g^{-1}t_j), w' \rangle \right|^p d\mu_G(g) = \Delta(t_i^{-1}) \int_H \Delta(h^{-1}) \left| \langle f(t_i h), w' \rangle \right|^p d\mu_H(h) \\ &= \Delta(t_i^{-1}) \int_H \left| \langle \pi(h)w, w' \rangle \right|^p d\mu_H(h) = \Delta(t_i^{-1}) \|\pi_{w,w'}\|_p^p < \infty. \end{aligned}$$

It follows that  $\tilde{\pi}$  is an  $L^p$ -representation.  $\square$

In general, it may happen that the induced representation has better integrability properties than the representation from which one induces, in the sense that the induced representation of an  $L^p$ -representation may be an  $L^q$ -representation for some  $q < p$  (even if one considers the “optimal” value of  $p$ ). Indeed, this for example happens if one induces  $L^p$ -representations of  $\mathbb{F}_2$  for large  $p$  to higher-rank simple Lie groups, since for every higher rank simple Lie group

## I.2. Application to representations of almost automorphism groups of trees

$G$ , there exists  $p(G) < \infty$  (depending on the group) such that every nontrivial unitary irreducible representation of  $G$  is an  $L^{p(G)}$ -representation (See Théorème 2.4.2 in [Cow79]). This phenomenon is not well understood and we do not know whether it can actually happen in the case of induction from an open subgroup.

The main point of this note is to show that in a specific case, namely for representations of a locally compact group  $G$  induced from representations of an open subgroup  $H$  with the Kunze–Stein property, we know that the (optimal)  $L^p$ -integrability of a representation is preserved under induction. A locally compact group  $G$  has the Kunze–Stein property if for every  $p \in [1, 2)$ , the convolution product on  $C_c(G)$  extends to a bounded bilinear map  $L^p(G) \times L^2(G) \rightarrow L^2(G)$ . This property originated from the work of Kunze and Stein [KS60], who proved it for  $\mathrm{SL}(2, \mathbb{R})$ .

**Theorem I.1.3.** *Let  $G$  be a locally compact group,  $H$  an open subgroup of  $G$ , and  $p \in [1, \infty)$ . Let  $(\pi, V)$  be a unitary representation of  $H$  and  $(\tilde{\pi}, \tilde{V})$  the associated induced representation of  $G$ . If  $H$  has the Kunze–Stein property, then  $\pi$  is an  $L^{p+}$ -representation of  $H$  if and only if  $\tilde{\pi}$  is an  $L^{p+}$ -representation of  $G$ .*

*Proof.* One direction is Proposition I.1.2. For the other direction, suppose that  $\tilde{\pi}$  is an  $L^{p+}$ -representation of  $G$ . Then  $\tilde{\pi}|_H$  (restriction as a function) is an  $L^{p+}$ -representation of  $H$ . Because  $\pi$  is contained in  $\tilde{\pi}|_H$  and  $H$  has the Kunze–Stein property, it follows from Proposition 4.3 in [LS21] that  $\pi$  is itself an  $L^{p+}$ -representation.  $\square$

If we do not assume  $H$  to have the Kunze–Stein property, the restriction  $\tilde{\pi}|_H$  might theoretically have better integrability properties than being an  $L^{p+}$ -representation. It would be interesting to have a better understanding of the relation between  $L^{p+}$ -representations of a group and  $L^{p+}$ -representations of its subgroups (also in the case of closed subgroups).

## I.2 Application to representations of almost automorphism groups of trees

For  $d \geq 3$ , let  $T_d$  be a  $d$ -regular tree and  $\mathrm{Aut}(T_d)$  the automorphism group of  $T_d$ , which carries a natural non-discrete, totally disconnected, locally compact group topology. In [Ner93], Neretin introduced the group  $N_d$  of almost automorphisms of  $T_d$ . The Neretin groups  $N_d$  also carry a natural non-discrete, totally disconnected, locally compact group topology, and  $\mathrm{Aut}(T_d)$  embeds into  $N_d$  as an open subgroup. It is known that the groups  $N_d$  are simple [Kap99] and that they do not have any lattices [BCGM12]. Recently, the Neretin groups have attracted increasing interest, also from the point of view of unitary representation theory (see e.g. [Ner19]). Notably, it was recently shown that the Neretin groups are not of type I [CLBMB22] (see also [Ari22]).

More generally, let  $H$  be a non-compact, closed subgroup of  $\mathrm{Aut}(T_d)$  acting transitively on the boundary  $\partial T_d$  of  $T_d$ . Then  $H$  satisfies the Kunze–Stein property [Neb88]. In Section 2 of [Led19], Lederle constructs the group  $\mathcal{F}(H)$  of  $H$ -honest

almost automorphisms, i.e. almost automorphisms of  $T_d$  that “locally look like” elements of  $H$ . She also shows that this group can be viewed as the topological full group of  $H$  acting on  $\partial T_d$ . If it is moreover assumed that  $H$  satisfies Tits’ independence property, then the group  $\mathcal{F}(H)$  carries a unique non-discrete, totally disconnected, locally compact group topology such that  $H$  embeds into  $\mathcal{F}(H)$  as an open subgroup (see Proposition 2.22 in [Led19]). A special class of such groups  $\mathcal{F}(H)$  is the class of coloured Neretin groups.

The unitary representation theory of  $\text{Aut}(T_d)$  goes back to [Ols77] (see also [FN91]), and more generally, for subgroups  $H$  of  $\text{Aut}(T_d)$  as above, it was described by Amann [Ama03]. It follows from Theorem I.1.3 and the above discussion that every unitary representation of  $H$  can be induced to  $\mathcal{F}(H)$  retaining the optimal  $L^{p+}$ -integrability of the representation.

We now make this more precise for the spherical complementary series of unitary representations under the additional assumption that the action of  $H$  on  $T_d$  is transitive. From the point of view of  $L^{p+}$ -representations, this part of the representation theory is most interesting. Indeed, for the groups  $H$  under consideration, all non-spherical unitary irreducible representations are tempered (i.e. they are  $L^{2+}$ -representations).

Let  $H$  be a non-compact, closed subgroup of  $\text{Aut}(T_d)$  acting transitively on  $T_d$  and on  $\partial T_d$  and satisfying Tits’ independence property. In what follows, we will use Amann’s description from Theorem 2 in [Ama03] of the unitary dual and refer to that text for details. The unitary irreducible spherical representations are in one-to-one correspondence with the positive-definite spherical functions, and because the latter are diagonal matrix coefficients with respect to a cyclic vector,  $L^p$ -integrability (for some  $p$ ) of the spherical function implies that the corresponding representation is an  $L^p$ -representation. As follows from (the proof of) Theorem 2 in [Ama03], the positive-definite spherical functions associated with non-tempered representations are parametrised by the set  $\mathcal{P} = \left[0, \frac{1}{2}\right) + i \left\{0, \frac{\pi}{\log(d-1)}\right\}$  through a bijection

$$\mathcal{P} \rightarrow \mathcal{SP}(K \backslash G / K), \quad z \mapsto \varphi_{\gamma_o(z)}.$$

Let  $\pi_{\gamma_o(z)}$  denote the (unique) spherical unitary irreducible representation associated with  $\varphi_{\gamma_o(z)}$ . For an explicit description of the spherical function  $\varphi_{\gamma_o(z)}$ , we refer to [Ama03]; we just recall its  $L^p$ -integrability. Outside the setting of Lie groups, the  $L^p$ -integrability of spherical functions was first considered for spherical functions on the free group  $\mathbb{F}_n$  (see Corollary of Theorem 2 in [FP82]). By the  $K$ -bi-invariance of spherical functions, this leads to the following consequence in the case of a subgroup  $H < \text{Aut}(T_d)$  as above: For  $p \in (2, \infty)$  and  $p' \in (1, 2)$  with  $\frac{1}{p} + \frac{1}{p'} = 1$ , the representation  $\pi_{\gamma_o(z)}$  is an  $L^{p+}(G)$ -representation if and only if  $\text{Re } z \in \left[\frac{1}{p}, \frac{1}{p'}\right]$ . We also refer to Lemma 4.11 in [HdLS20] for a direct proof of this fact.

**Theorem I.2.1.** *Let  $H$  be a non-compact, closed subgroup of  $\text{Aut}(T_d)$  acting transitively on  $T_d$  and on  $\partial T_d$  and satisfying Tits’ independence property. Let  $p \in (2, \infty)$  and  $p' \in (1, 2)$  be such that  $\frac{1}{p} + \frac{1}{p'} = 1$ . Then the representation of  $\mathcal{F}(H)$  induced from  $\pi_{\gamma_o(z)}$  is an  $L^{p+}$ -representation if and only if  $\text{Re } z \in \left[\frac{1}{p}, \frac{1}{p'}\right]$ .*

As a consequence, for every  $p \in (2, \infty)$ , the group  $\mathcal{F}(H)$  has a unitary  $L^{p+}$ -representation that is not an  $L^{q+}$ -representation for any  $q < p$ . This gives an uncountable family of pairwise inequivalent non-tempered unitary representations of  $\mathcal{F}(H)$ .

A version of the above theorem can also be formulated if the action of  $H$  on  $T_d$  is not transitive. However, due to an inaccuracy in the description of the unitary dual in [Ama03] in that case (as pointed out in e.g. [Sem23]), the parametrisation of the spherical complementary series of  $H$  requires some more care.

### I.3 Remark on $L^{p+}$ -group- $C^*$ -algebras

In the previous section, we constructed  $L^{p+}$ -representations of  $\mathcal{F}(H)$  that are not  $L^{q+}$ -representations for  $q < p$ . It is interesting to point out that by  $C^*$ -algebraic methods, it follows that the  $L^{p+}$ -group- $C^*$ -algebras  $C_{L^{p+}}^*(\mathcal{F}(H))$  of  $\mathcal{F}(H)$  must be pairwise canonically distinct for  $p \in [2, \infty)$ . More precisely, since  $H$  is an open subgroup of  $\mathcal{F}(H)$ , it follows from Proposition I.1.2, the end of Section 3 in [Wie15] and Theorem A in [HdLS20] that, for every  $2 \leq q < p < \infty$ , the canonical quotient map  $C_{L^{p+}}^*(\mathcal{F}(H)) \rightarrow C_{L^{q+}}^*(\mathcal{F}(H))$  is not injective. Although this gives some information about the existence of  $L^{p+}$ -representations, this method does not give information about the optimal  $L^{p+}$ -integrability of specific representations (such as induced representations).

**Acknowledgements** We thank Pierre-Emmanuel Caprace and Sven Raum for useful comments and for pointing out [Ari22]. We also thank Pierre-Emmanuel Caprace for pointing out the inaccuracy in [Ama03].

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# Paper II

## Property (T) for Banach algebras

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Accepted for publication in *Journal of Operator Theory*.

**Abstract** We define and study the notion of property (T) for Banach algebras, generalizing the one from  $C^*$ -algebras. For a second countable locally compact group  $G$  and a given family of Banach spaces  $\mathcal{E}$ , we prove that our Banach algebraic property  $(T_{\mathcal{E}})$  of the symmetrized pseudofunction algebras  $F_{\mathcal{E}}^*(G)$  characterizes the Banach property  $(T_{\mathcal{E}})$  of Bader, Furman, Gelfand and Monod for groups. In case  $G$  is a discrete group and  $\mathcal{E}$  is the class of  $L^p$ -spaces for  $1 \leq p < \infty$ , we also achieve the analogue characterization using the symmetrized  $p$ -pseudofunction algebras  $F_{\lambda_p}^*(G)$ .

### II.1 Introduction

In 1967 David Kazhdan introduced in [Kaz67] the notion of property (T) for groups in order to prove finite generation of lattices in higher rank Lie groups. Today Kazhdan's property (T) is a central notion of analytic group theory being used in numerous proofs including Margulis superrigidity theorem. A second countable locally compact group  $G$  has property (T) if, whenever a unitary representation of  $G$  contains a net of almost invariant vectors, it has a non-zero invariant vector. Lattices in higher rank semisimple Lie groups, as well as lattices in  $Sp(1, n)$  enjoy this property. We refer the reader to [BLV08] for a comprehensive treatment of Kazhdan's property (T).

Classically, property (T) concerns unitary representations and the class of complex Hilbert spaces. In [BFGM07], Bader, Furman, Gelfand and Monod extended this notion to isometric representations on Banach spaces. They showed

that Kazhdan's property (T) is equivalent to their Banach algebraic property  $(T_{L^p})$  for  $1 \leq p < \infty$ .

On the operator algebraic side, the notion of property (T) was brought to von Neumann algebras by Connes in [Con80] for type  $II_1$ -factors and later by Connes and Jones in [CJ85] for general von Neumann algebras, where unitary representations were replaced by bimodules (or Connes correspondences). Turning to  $C^*$ -algebras, Bekka adopted Connes's definition and formulated the notion of property (T) for unital  $C^*$ -algebras in [Bek05] and Ng defined two versions of property (T) for general  $C^*$ -algebras in [Ng13]. Although property (T) for  $C^*$ -algebras is a younger topic compared with von Neumann algebras, it already proved useful in applications. For example, connections with nuclearity (i.e., amenability) of  $C^*$ -algebras, Haagerup property for  $C^*$ -algebras, and property (T) of quantum groups were studied respectively in [Bro06], in [Suz13] and [Men17], and in [KS12].

In this article we generalize the notion of property (T) to Banach algebras that possess a bounded approximate unit. Our aim is to characterize our Banach algebraic property  $(T_{\mathcal{E}})$  in terms of the group theoretic property  $(T_{\mathcal{E}})$  of Bader, Furman, Gelander and Monod [BFGM07]. Our definition, which we state below, adopts the stronger version of property (T) of Ng in Definition 2.1 in [Ng13] as it is stated in section 2 of [BN19]. Note that our terminology is different from that of Bekka and Ng; see Warning II.3.

**Definition A.** (Definition II.3.1) Let  $\mathcal{A}$  be a Banach algebra with a bounded approximate unit and let  $\mathcal{E}$  be a class of Banach spaces. We say that  $\mathcal{A}$  has property  $(T_{\mathcal{E}})$  if, whenever  $E \in \mathcal{E}$  is an essential  $\mathcal{A}$ -bimodule admitting a strictly almost  $\mathcal{A}$ -central net of unit vectors  $(\xi_i)_{i \in I}$  then there exists a net of central vectors  $(\eta_i)_{i \in I}$  in  $E$  such that  $\|\xi_i - \eta_i\|_E \rightarrow 0$ .

We further define a weaker version of this property (see Definition II.3.2), where we only require the existence of a non-zero central vector. The weak version is referred to as *weak property*  $(T_{\mathcal{E}})$  and it is an extension of the weaker version of property (T) of Bekka and Ng for  $C^*$ -algebras (see Definition 6 in [Bek05] for the unital case and Definition 2.1 in [Ng13] for the general case). When  $\mathcal{A}$  is unital, our weak property  $(T_{\mathcal{E}})$  coincides with the definition suggested by Bekka in Remark 18 of [Bek05] of a Banach space version of property (T) for arbitrary normed algebras. When  $\mathcal{A}$  is a  $C^*$ -algebra and we consider the class of complex Hilbert spaces, our definitions coincide with Ng's [Ng13] in the general case, and with Bekka's [Bek05] in the unital case.

Given a locally compact group  $G$ , we consider Banach algebras constructed from actions on a family of Banach spaces  $\mathcal{E}$  referred to as pseudofunction algebras  $F_{\mathcal{E}}(G)$ . We shall consider both the pseudofunction algebra  $F_{\mathcal{E}}(G)$  and its symmetrized version  $F_{\mathcal{E}}^*(G)$ . See paragraph II.2.3 in the preliminaries for a definition. We are particularly interested in the symmetrized pseudofunction algebra associated with the class of representations on  $L^p$ -spaces,  $F_{L^p}^*(G)$ , as well as the symmetrized version of the  $p$ -pseudofunction algebra,  $F_{\lambda_p}^*(G)$ . Pseudofunction algebras were first studied by Herz already in the 1970's [Her71], and they have subsequently been studied intensely in the context of harmonic analysis. Later they appeared in the work of Phillips, who provided an operator algebraic ground

for studying these Banach algebras (see, e.g., [Phi13], [PV20], and [BP19].) The symmetrized version of the  $p$ -pseudofunction algebra was first considered by Kasparov and Yu in [KY], and were studied in [LY17] in the context of the Baum-Connes conjecture, in [SW20] in the context of quasi-Hermitian groups as well as in [SW24] in the context of exotic group  $C^*$ -algebras.

The main results of this article characterizes our Banach algebraic property  $(T_{\mathcal{E}})$  for symmetrized pseudofunction algebras in terms of Bader *et al*'s property  $(T_{\mathcal{E}})$ , for a general class of Banach spaces  $\mathcal{E}$  and, in particular, for the class  $L^p$  consisting of  $L^p$ -spaces on  $\sigma$ -finite measure spaces. We state the results here for the symmetrized pseudofunction algebras, but they hold for the non-symmetrized versions, as well.

**Theorem B.** *(Theorem II.3.10A and II.3.10B) Let  $G$  be a locally compact group and let  $\mathcal{E}$  be a class of Banach spaces. The following are equivalent:*

- (i)  $G$  has (weak) property  $(T_{\mathcal{E}})$ ,
- (ii)  $F_{\mathcal{E}}^*(G)$  has (weak) property  $(T_{\mathcal{E}})$ .

Under the assumption that  $\mathcal{E}$  is the class  $L^p$ -spaces, Theorem B implies:

**Corollary C.** *(Corollary II.3.14) Let  $G$  be a second countable locally compact group with property  $(T)$  and let  $1 \leq p \leq 2$ . Then  $F_{L^p}^*(G)$  has property  $(T_{L^q})$ , for all  $1 \leq q \leq p$  and all  $p' \leq q < \infty$ , where  $p'$  is the Hölder conjugate of  $p$ .*

Assuming further that the group is discrete, Theorem B implies:

**Corollary D.** *(Corollary II.3.15) Let  $\Gamma$  be a discrete group with property  $(T)$  and let  $1 \leq p \leq 2$ . Then  $F_{L^p}^*(\Gamma)$  has property  $(T_{L^q})$ , for all  $1 \leq q < \infty$ .*

Continuing with the assumptions that  $\Gamma$  is a discrete group and  $\mathcal{E}$  is the class  $L^p$ -spaces, we obtain, in addition to the characterization from Theorem B, a characterization of property  $(T_{L^p})$  in terms of the symmetrized  $p$ -pseudofunction algebra  $F_{\lambda_p}^*(\Gamma)$ .

**Theorem E.** *(Theorem II.4.4) Let  $\Gamma$  be a discrete group and let  $1 \leq p < \infty$ . The following are equivalent:*

- (i)  $\Gamma$  has property  $(T_{L^p})$ ,
- (ii)  $F_{L^p}^*(\Gamma)$  has property  $(T_{L^p})$ ,
- (iii)  $F_{\lambda_p}^*(\Gamma)$  has property  $(T_{L^p})$ .

Theorem E holds for the non-symmetrized versions of the algebras, as well. The proof relies on an  $L^p$ -analogue of Fell's absorption principle; see Lemma II.4.1.

This paper is organized as follows. Section 2 contains preliminaries on actions of Banach algebras, multiplier algebras, symmetrized pseudofunction algebras as well as property  $(T_{\mathcal{E}})$  for actions of groups on Banach spaces. In section 3 we prove Theorem B and its implications Corollary C and Corollary D. In section 4 we prove Theorem E. Further, we show that weak property  $(T_{SL^p})$  is stronger than property  $(T_{L^p})$  for discrete groups.

## II.2 Preliminaries

### II.2.1 Actions of Banach algebras on Banach spaces

For a Banach algebra  $\mathcal{A}$ , we denote by  $\mathcal{A}^{\text{op}}$  its opposite algebra, i.e., the Banach algebra with the same underlying Banach space as  $\mathcal{A}$  but with multiplication in reversed order.

**Definition II.2.1.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras and  $E$  a Banach space. A *left action* of  $\mathcal{A}$  on  $E$  is a contractive representation of  $\mathcal{A}$  on  $E$ . A *right action* of  $\mathcal{A}$  on  $E$  is a contractive representation of  $\mathcal{A}^{\text{op}}$  on  $E$ . We say that  $E$  is a *left (right)  $\mathcal{A}$ -module* if it carries a left (right) action of  $\mathcal{A}$ . Further,  $E$  is an  *$\mathcal{A}$ - $\mathcal{B}$ -bimodule* if it carries a left action  $\varphi$  of  $\mathcal{A}$  and a right action  $\psi$  of  $\mathcal{B}$  with commuting ranges. We write

$$a \cdot \xi \cdot b = \psi(b)\varphi(a)\xi, \quad \text{for } a \in \mathcal{A}, b \in \mathcal{B} \text{ and } \xi \in E.$$

An  $\mathcal{A}$ - $\mathcal{A}$ -bimodule is simply referred to as an  $\mathcal{A}$ -bimodule.

**Definition II.2.2.** An  $\mathcal{A}$ - $\mathcal{B}$ -bimodule  $E$  is said to be *essential* if the span of  $\mathcal{A} \cdot E \cdot \mathcal{B}$  is dense in  $E$ . Further,  $E$  is said to be *faithful* if, whenever  $\xi \in E$  satisfies  $a \cdot \xi \cdot b = 0$ , for all  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ , then  $\xi = 0$ .

*Remark II.2.3.* Let  $\mathcal{A}$  be a Banach algebra with a bounded approximate unit  $(u_i)_i$ . Then any essential  $\mathcal{A}$ -bimodule will necessarily be faithful. Indeed, if  $E$  is an essential  $\mathcal{A}$ -bimodule it follows from Cohen's factorization theorem [Coh59] that  $E$  is pseudo-unital, i.e.,  $E = \mathcal{A} \cdot E \cdot \mathcal{A}$ . Suppose  $\xi \in E$  and  $a, b \in \mathcal{A}$  are such that  $ca \cdot \xi \cdot bd = 0$ , for all  $c, d \in \mathcal{A}$ , then  $a \cdot \xi \cdot b = 0$ . Then, for each  $i$ ,

$$\begin{aligned} \|a \cdot \xi \cdot b\| &= \|a \cdot \xi \cdot b - u_i a \cdot \xi \cdot b\| + \|u_i a \cdot \xi \cdot b - u_i a \cdot \xi \cdot b u_i\| \\ &\leq \|a - u_i a\| \|\xi \cdot b\| + \sup_j \|u_j\| \|a \cdot \xi\| \|b - b u_i\|. \end{aligned}$$

As the right-hand side tends to zero in  $i$ , we see that  $a \cdot \xi \cdot b = 0$ . Since all elements of  $E$  admits a factorization in the form  $a \cdot \xi \cdot b$ , it follows that  $E$  is faithful.

The Banach  $*$ -algebra  $L^1(G)$ , for a locally compact group  $G$ , plays an important role among the Banach algebras we consider. Since  $L^1(G)$  always carries a bounded approximate unit, it is essential and faithful as a bimodule over itself. The following fact connecting its contractive representation theory with the isometric representation theory of  $G$  is folklore:

**Proposition II.2.4.** *Let  $G$  be a locally compact group and  $E$  a Banach space. There is a 1-1 correspondence between non-degenerate, contractive representations of  $L^1(G)$  on  $E$  and strongly continuous isometric representations of  $G$  on  $E$ .*

The Banach  $*$ -algebra  $L^1(G)$  is in the following precise sense its own opposite, as may be easily verified:

**Proposition II.2.5.** *The map  $\tilde{\square} : L^1(G) \rightarrow L^1(G)^{\text{op}}$  given by*

$$\tilde{f}(s) = \Delta(s^{-1})f(s^{-1}), \quad \text{for } s \in G, \quad (\text{II.1})$$

*is an isometric  $*$ -isomorphism.*

## II.2.2 Multipliers of Banach algebras

We refer the reader to Section 2 in [Daw10] for details and more results on multipliers of modules.

**Definition II.2.6.** Let  $\mathcal{A}$  be a Banach algebra. A *multiplier* of  $\mathcal{A}$  is a pair of maps  $(L, R)$  from  $\mathcal{A}$  to itself satisfying

$$aL(b) = R(a)b, \quad \text{for all } a, b \in \mathcal{A}.$$

We denote by  $M(\mathcal{A})$  the set of all multipliers on  $\mathcal{A}$ . This is a linear space with addition and scalar multiplication defined in the obvious way.

There is a canonical linear map  $\mathcal{A} \rightarrow M(\mathcal{A})$  defined by assigning to each  $a_0 \in \mathcal{A}$  a pair of maps  $L_{a_0}, R_{a_0} : \mathcal{A} \rightarrow \mathcal{A}$  as follows:

$$L_{a_0} : a \mapsto a_0a \quad \text{and} \quad R_{a_0} : a \mapsto aa_0, \quad \text{for } a \in \mathcal{A}.$$

If  $\mathcal{A}$  is faithful as a bimodule over itself, this map is injective. If  $\mathcal{A}$  is unital, it is surjective. In general, it need neither be injective nor surjective.

When  $\mathcal{A}$  is faithful as a bimodule over itself, one can show that  $M(\mathcal{A})$  embeds linearly into  $\mathcal{B}(\mathcal{A}) \oplus_\infty \mathcal{B}(\mathcal{A})$ . Hence,  $M(\mathcal{A})$  inherits the *strict topology*, i.e., the locally convex topology generated by the family of seminorms  $(L, R) \mapsto \|L(a)\| + \|R(a)\|$ , where  $a \in \mathcal{A}$ . The next proposition may be verified with routine arguments:

**Proposition II.2.7.** *Let  $\mathcal{A}$  be a Banach algebra and assume that  $\mathcal{A}$  is faithful as a bimodule over itself. Then  $M(\mathcal{A})$  is a strictly closed subspace of  $\mathcal{B}(\mathcal{A}) \oplus_\infty \mathcal{B}(\mathcal{A})$ .*

Lemma II.2.8 concerning strictly compact subsets of  $M(\mathcal{A})$  is Lemma 10 in [BN19] put in our more general setting. The proof is the same and so we omit it.

**Lemma II.2.8.** *Let  $\mathcal{A}$  be a Banach algebra which is faithful as a bimodule over itself and let  $S$  be a non-empty strictly compact subset of  $M(\mathcal{A})$ . Then  $S$  satisfies the following two properties:*

- (i)  $S$  is norm-bounded,
- (ii) for any element  $a_0 \in \mathcal{A}$  and any  $\varepsilon > 0$ , there exist a finite number of elements  $x_1, \dots, x_n \in S$  such that, for every  $x \in S$ , there is a  $k \in \{1, \dots, n\}$  for which  $\|x \cdot a_0 - x_k \cdot a_0\|_{\mathcal{A}} < \varepsilon$ .

Let  $\mathcal{A}$  be a Banach algebra with a bounded approximate unit so that any essential  $\mathcal{A}$ -bimodule is automatically faithful. Given another Banach algebra  $\mathcal{B}$  and a bounded homomorphism  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ , the Banach algebra  $\mathcal{B}$  becomes in a natural way a bimodule over  $\mathcal{A}$ . Moreover, if  $\varphi$  has dense range, the image of any bounded approximate unit on  $\mathcal{A}$  is a bounded approximate unit on  $\mathcal{B}$ . The following is a special case of Theorem 2.8 in [Daw10]:

**Theorem II.2.9.** *Let  $\mathcal{A}$  be a Banach algebra with a bounded approximate unit, let  $\mathcal{B}$  be another Banach algebra and let  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  be a bounded homomorphism with dense range. There is a unique extension  $\Phi : M(\mathcal{A}) \rightarrow M(\mathcal{B})$  and this extension is strictly continuous.*

We close this part with some remarks on the multiplier algebra of  $L^1(G)$ , for a locally compact group  $G$ . For  $s \in G$ , we denote by  $L_s$  and  $R_s$  the left, respectively, right translation operators on  $L^1(G)$ . Precisely, for  $s, t \in G$  and  $f \in L^1(G)$ ,

$$L_s f(t) = f(s^{-1}t), \quad \text{and} \quad R_s f(t) = f(ts).$$

Then  $(L_s, \Delta(s^{-1})R_{s^{-1}})$  is a multiplier of  $L^1(G)$ . This gives rise to a multiplicative embedding  $G \hookrightarrow M(L^1(G))$ , and this embedding is continuous when  $M(L^1(G))$  is equipped with the strict topology. Thus, given an essential  $L^1(G)$ -bimodule  $E$ , the group  $G$ , respectively, its opposite  $G^{\text{op}}$  act on  $E$  via the extension of the bimodule structure to the multiplier algebra. Proposition II.2.10, which may be verified with a straight forward computation, shows that these actions agree with the actions we already have from the  $L^1(G)$ -bimodule structure via Proposition II.2.4.

**Proposition II.2.10.** *Let  $G$  be a locally compact group and let  $E$  be an essential  $L^1(G)$ -bimodule with left action  $\varphi$  and right-action  $\psi$ . For each  $\xi \in E$  and  $s \in G$ ,*

$$(L_s, \Delta(s^{-1})R_{s^{-1}}) \cdot \xi = \varphi(s)\xi \quad \text{and} \quad \xi \cdot (L_s, \Delta(s^{-1})R_{s^{-1}}) = \psi(s)\xi.$$

### II.2.3 Symmetrized pseudofunction algebras

Let  $G$  be a locally compact group. Given a class of Banach spaces  $\mathcal{E}$ , denote by  $\text{Rep}_{\mathcal{E}}(G)$  the class of strongly continuous isometric representations of  $G$  on a Banach space in  $\mathcal{E}$ . For a subclass  $\mathcal{R}$  of  $\text{Rep}_{\mathcal{E}}(G)$ , we define a seminorm on  $L^1(G)$  by setting

$$\|f\|_{\mathcal{R}} = \sup \{ \|\pi(f)\| \mid \pi \in \mathcal{R} \}.$$

Set  $I_{\mathcal{R}} = \bigcap_{\pi \in \mathcal{R}} \ker(\pi)$ . This is a closed 2-sided ideal in  $L^1(G)$ , and so, the quotient  $L^1(G)/I_{\mathcal{R}}$  inherits the algebra structure from  $L^1(G)$ . We denote by  $F_{\mathcal{R}}(G)$  the completion of  $L^1(G)/I_{\mathcal{R}}$  with respect to the norm induced by  $\|\cdot\|_{\mathcal{R}}$ . This is a Banach algebra with multiplication extending the convolution product on  $L^1(G)$ ; we refer to it as the *Banach algebra of  $\mathcal{R}$ -pseudofunctions*. When  $\mathcal{R}$  is all of  $\text{Rep}_{\mathcal{E}}(G)$ , we shall denote by  $F_{\mathcal{E}}(G)$  the resulting Banach algebra. When  $\mathcal{R}$  consists of only one representation, say  $\pi$ , we simply write  $F_{\pi}(G)$ . Accordingly, we refer to these Banach algebras as algebras of  *$\mathcal{E}$ -pseudofunctions*, respectively,  *$\pi$ -pseudofunctions*.

Well-known examples of pseudofunction algebras include the universal and the reduced group  $C^*$ -algebras,  $C^*(G)$  and  $C_r^*(G)$ , respectively. In the notation introduced above, the former is the pseudofunction algebra  $F_{\mathcal{H}}(G)$ , where  $\mathcal{H}$  is the class of complex Hilbert spaces, and the latter is  $F_{\lambda}(G)$ , where  $\lambda$  the left-regular representation of  $G$ . Further, for  $1 \leq p < \infty$  and  $\lambda_p$  the left-regular representation of  $G$  on  $L^p(G)$ ,  $F_{\lambda_p}(G)$  is the Banach algebra of  $p$ -pseudofunctions, which goes back to work of Herz and it is often denoted by  $PF_p(G)$ . This Banach algebra also appeared in work of Phillips, e.g. [Phi13], where it is denoted by  $F_r^p(G)$  to emphasize its connection to the reduced group  $C^*$ -algebra.

*Remark II.2.11.* It is easy to see that if  $\pi$  is an isometric representation of  $G$ , and  $\pi$  lies in the class  $\mathcal{R}$ , then  $\pi$  extends to a non-degenerate contractive representation of  $F_{\mathcal{R}}(G)$ . That is,  $F_{\mathcal{R}}(G)$  is universal for  $\mathcal{R}$  in the same way that  $C^*(G)$  is universal for all unitary representations of  $G$ . Conversely, by Proposition II.2.4, if  $\pi$  is a non-degenerate contractive representation of  $F_{\mathcal{R}}(G)$  then  $\pi$  is the extension of an integrated form of an isometric representation of  $G$ . However, we are not guaranteed that  $\pi$  lies in the class  $\mathcal{R}$ .

The involution on  $L^1(G)$  need not extend to  $F_{\mathcal{R}}(G)$ , and so,  $F_{\mathcal{R}}(G)$  is in general only a Banach algebra and not necessarily a Banach \*-algebra. However, if the class  $\mathcal{R}$  is closed under duality, the involution on  $L^1(G)$  does extend. Recall that if  $\pi$  is an isometric representation of  $G$  on a Banach space  $E$ , its dual representation  $\pi^*$  is the isometric representation on the dual Banach space  $E^*$  given, for  $t \in G$ ,  $\eta \in E^*$  and  $x \in E$ , by

$$(\pi^*(t)\eta)(x) = \eta(\pi(t^{-1})x).$$

We say that the class  $\mathcal{R}$  is closed under duality if  $\pi^* \in \mathcal{R}$  whenever  $\pi \in \mathcal{R}$ . Proposition II.2.12 below is proven in a special case in Proposition 4.2 in [SW20]. The proof in the generality stated here is essentially the same, and so, we omit it.

**Proposition II.2.12.** *Let  $\mathcal{R}$  be a class of continuous isometric representations of  $G$  closed under duality. Then the involution on  $L^1(G)$  is an isometry with respect to the norm induced by  $\mathcal{R}$ .*

For a class  $\mathcal{R}$  of strongly continuous isometric representations of  $G$ , denote by  $\mathcal{R}^*$  the smallest class of strongly continuous isometric representations of  $G$  which is closed under duality and which contains  $\mathcal{R}$ . We denote by  $F_{\mathcal{R}^*}^*(G)$  the completion of  $L^1(G)/I_{\mathcal{R}^*}$  with respect to the norm

$$\|f\|_{F_{\mathcal{R}^*}^*(G)} = \sup \{ \|\pi(f)\| \mid \pi \in \mathcal{R}^* \}.$$

By Proposition II.2.12,  $F_{\mathcal{R}^*}^*(G)$  is a Banach \*-algebra, and we shall refer to it as the *symmetrized Banach \*-algebra of  $\mathcal{R}$ -pseudofunctions*. As in the non-symmetrized setting, when  $\mathcal{R}$  is all of  $\text{Rep}_{\mathcal{E}}(G)$  or when  $\mathcal{R}$  consists of a single representation  $\pi$ , we shall write  $F_{\mathcal{E}}^*(G)$ , respectively,  $F_{\pi}^*(G)$ , and we refer to these accordingly.

*Remark II.2.13.* Let  $\mathcal{E}_{\text{ref}}$  be the class of all reflexive Banach spaces. For a subclass  $\mathcal{E} \subset \mathcal{E}_{\text{ref}}$ , denote by  $\mathcal{E}'$  the class consisting of the Banach spaces which are dual to the Banach spaces in  $\mathcal{E}$ . Let  $\mathcal{R}$  be a subclass of  $\text{Rep}_{\mathcal{E}}(G)$  and denote by  $\mathcal{R}'$  the subclass of  $\text{Rep}_{\mathcal{E}'}(G)$  consisting of representations which are dual to the representations in  $\mathcal{R}$ . Then  $\mathcal{R}^* = \mathcal{R} \cup \mathcal{R}'$ . In this case, a straight forward computation gives

$$\|f\|_{F_{\mathcal{R}^*}^*(G)} = \max \left\{ \|f\|_{F_{\mathcal{R}}(G)}, \|\tilde{f}\|_{F_{\mathcal{R}}(G)} \right\}.$$

*Remark II.2.14.* Similar to  $L^1(G)$ ,  $F_{\mathcal{R}^*}^*(G)$  is self-opposite via the map defined in equation (II.1). This need not be true for general pseudofunction algebras.

## II.2.4 Property (T) for groups acting on Banach spaces

For a Banach space  $E$ , denote by  $\text{Isom}(E)$  the group of surjective linear isometries on  $E$ . A strongly continuous isometric representation of a locally compact group  $G$  on a Banach space  $E$  is a strongly continuous group homomorphism  $\pi : G \rightarrow \text{Isom}(E)$ . Given such a representation  $(\pi, E)$ , we denote by  $E^\pi$  the subspace of  $G$ -invariant vectors. In [BFGM07], Bader, Furman, Gelander and Monod define property  $(T_\mathcal{E})$  as follows:

**Definition II.2.15.** Definition 1.1 in [BFGM07] Let  $\mathcal{E}$  be a class of Banach spaces. A locally compact group  $G$  has property  $(T_\mathcal{E})$  if, for any continuous isometric representation  $(\pi, E)$  with  $E$  in the class  $\mathcal{E}$ , the quotient representation  $\pi' : G \rightarrow \text{Isom}(E/E^\pi)$  does not have almost  $G$ -invariant vectors. If  $\mathcal{E}$  consists of a single Banach space  $E$ , we write  $(T_E)$  instead of  $(T_\mathcal{E})$ .

*Remark II.2.16.* We recall from Theorem A in [BFGM07] that, for a second countable locally compact group  $G$ , Kazhdan's property (T) coincides with property  $(T_{L^p(\mu)})$ , for any  $\sigma$ -finite measure  $\mu$  and any  $1 \leq p < \infty$ .

We shall use the following alternative definition of property  $(T_\mathcal{E})$ , which is equivalent to the definition above by Lemma 18 in [Tan17]. There, the lemma is stated for second countable locally compact groups, but the additional assumption that the group is second countable can be dropped.

**Definition II.2.17.** Let  $\mathcal{E}$  be a class of Banach spaces. A locally compact group  $G$  has *property*  $(T_\mathcal{E})$  if, whenever  $(\pi, E)$  is a strongly continuous isometric representation of  $G$  with  $E$  in the class  $\mathcal{E}$  admitting a net of almost invariant unit vectors  $(\xi_i)_{i \in I}$ , there exists a net of  $G$ -invariant vectors  $(\eta_i)_{i \in I}$  such that  $\|\xi_i - \eta_i\|_E \rightarrow 0$ .

It may be tempting to define the Banach space version of property (T) for groups parallel to the often used definition of Kazhdan's property (T) which only requires the existence of a non-zero invariant vector. A priori, this is a weaker property.

**Definition II.2.18.** Let  $\mathcal{E}$  be a class of Banach spaces. A locally compact group  $G$  has *weak property*  $(T_\mathcal{E})$  if any strongly continuous isometric representation  $(\pi, E)$  with  $E \in \mathcal{E}$  admitting almost invariant vectors has a non-zero  $G$ -invariant vector.

It is well-known that property  $(T_\mathcal{H})$  is equivalent to weak property  $(T_\mathcal{H})$  when  $\mathcal{H}$  is the class of complex Hilbert spaces, in which case we recover Kazhdan's property (T). A bit more generally, Proposition II.2.19 gives two sufficient conditions on the class  $\mathcal{E}$  for the equivalence of property  $(T_\mathcal{E})$  and weak property  $(T_\mathcal{E})$ . This may be of independent interest. The conditions are well-known to experts, but to our knowledge, they do not appear explicitly in the literature.

**Proposition II.2.19.** *For any second countable locally compact group  $G$  and any class of Banach spaces  $\mathcal{E}$ , property  $(T_\mathcal{E})$  implies weak property  $(T_\mathcal{E})$ . The converse is true if  $\mathcal{E}$  satisfies either one of the following properties:*

- (i)  $\mathcal{E}$  is stable under quotients,

(ii)  $\mathcal{E}$  is a class of superreflexive Banach spaces stable under taking complemented subspaces.

*Proof.* Assume  $G$  does not have property  $(T_{\mathcal{E}})$ . We can then find a continuous isometric representation  $(\pi, E)$  of  $G$  with  $E$  in  $\mathcal{E}$  such that the quotient  $E/E^{\pi}$  admits a net of almost invariant vectors. However,  $E/E^{\pi}$ , by construction, has no non-zero  $G$ -invariant vectors. If there is an isometric representation of  $G$  on a space  $F$  in  $\mathcal{E}$  and a bounded equivariant isomorphism from  $F$  to  $E/E^{\pi}$ , it follows that  $G$  does not have property  $(T_{\mathcal{E}})$ . This is trivially the case if  $\mathcal{E}$  is stable under quotients. If  $\mathcal{E}$  consists of superreflexive Banach spaces then  $E^{\pi}$  is a complemented subspace and its complement is isomorphic to the quotient  $E/E^{\pi}$  (see, e.g., Proposition 2.6 in [BFGM07]). If  $\mathcal{E}$  furthermore is stable under taking complemented subspaces, we have a contraction from a space in  $\mathcal{E}$  to the quotient  $E/E^{\pi}$ . Hence, if the class  $\mathcal{E}$  satisfies either of the conditions (i) or (ii), we see that if  $G$  does not have property  $(T_{\mathcal{E}})$  it does not have weak property  $(T_{\mathcal{E}})$  either.  $\square$

## II.3 Property $(T_{\mathcal{E}})$ for Banach algebras

In this section we define property  $(T_{\mathcal{E}})$ , as well as a weaker version of it, for a Banach algebra acting on a family of Banach spaces  $\mathcal{E}$ . Our definitions extend that of Bekka-Ng in [BN19] for (not necessarily unital)  $C^*$ -algebras to a Banach algebraic setting.

The main result in this section, Theorem II.3.10A, relates property  $(T_{\mathcal{E}})$  of Bader *et al* for locally compact groups to that of its symmetrized Banach  $*$ -algebra of  $\mathcal{E}$ -pseudofunctions. Denoting by  $L^p$  the class of all  $L^p$ -spaces on  $\sigma$ -finite measure spaces, we show that property  $(T_{L^p})$  for  $F_{L^p}^*(G)$  is independent on the parameter  $p$ . Finally, under the assumption that  $G$  has Kazhdan's property  $(T)$ , we obtain property  $(T_{L^q})$  for  $F_{L^p}^*(G)$ .

Let  $\mathcal{A}$  be a Banach algebra and  $E$  an  $\mathcal{A}$ -bimodule. We say that  $\xi \in E$  is  $\mathcal{A}$ -central if, for all  $a \in \mathcal{A}$ ,  $a \cdot \xi = \xi \cdot a$ . The set of all such elements constitutes a closed subspace of  $E$ , which we denote by  $E^{\mathcal{A}}$ . A net  $(\xi_i)_{i \in I}$  in  $E$  is said to be *almost  $\mathcal{A}$ -central* if, for every finite subset  $F \subset \mathcal{A}$  and every  $\varepsilon > 0$ , there is an index  $i_0 \in I$  such that, for all  $i \succcurlyeq i_0$ ,

$$\sup_{a \in F} \|a \cdot \xi_i - \xi_i \cdot a\|_E < \varepsilon.$$

A net  $(\xi_i)_{i \in I}$  in  $E$  is said to be *strictly almost  $\mathcal{A}$ -central* if, for every strictly compact subset  $S \subset M(\mathcal{A})$  and every  $\varepsilon > 0$ , there is an index  $i_0 \in I$  such that, for all  $i \succcurlyeq i_0$ ,

$$\sup_{x \in S} \|x \cdot \xi_i - \xi_i \cdot x\|_E < \varepsilon.$$

We can now state our two main definitions:

**Definition II.3.1.** Let  $\mathcal{A}$  be a Banach algebra with a bounded approximate unit and let  $\mathcal{E}$  be a class of Banach spaces. We say that  $\mathcal{A}$  has *property  $(T_{\mathcal{E}})$*  if,

whenever  $E \in \mathcal{E}$  is an essential  $\mathcal{A}$ -bimodule admitting a net of strictly almost  $\mathcal{A}$ -central unit vectors  $(\xi_i)_{i \in I}$ , then there exists a net  $(\eta_i)_{i \in I}$  of  $\mathcal{A}$ -central vectors such that  $\|\xi_i - \eta_i\|_E \rightarrow 0$ .

**Definition II.3.2.** Let  $\mathcal{A}$  be a Banach algebra with a bounded approximate unit and let  $\mathcal{E}$  be a class of Banach spaces. We say that  $\mathcal{A}$  has *weak property*  $(T_{\mathcal{E}})$  if, whenever  $E \in \mathcal{E}$  is an essential  $\mathcal{A}$ -bimodule admitting a net of strictly almost  $\mathcal{A}$ -central unit vectors, then  $E$  contains a non-zero  $\mathcal{A}$ -central vector.

It is immediately clear that property  $(T_{\mathcal{E}})$  implies weak property  $(T_{\mathcal{E}})$ .

*Remark II.3.3.* We have chosen to restrict the definitions of the two versions of property  $(T_{\mathcal{E}})$  to Banach algebras possessing a bounded approximate unit. This includes, in particular, all the pseudofunction algebras. The definitions, however, are sensible for any Banach algebra which is faithful as a bimodule over itself.

*Remark II.3.4.* When  $\mathcal{A}$  is a  $C^*$ -algebra and  $\mathcal{H}$  is the class of complex Hilbert spaces, our property  $(T_{\mathcal{H}})$  for  $\mathcal{A}$  recovers the stronger version of property (T) of Bekka and Ng while our weak property  $(T_{\mathcal{H}})$  recovers the weaker version of their property (T). Indeed, the assumption that the bimodules are essential forces the extension of the bimodule structure to the multiplier algebra to be unital, and unital contractive algebra homomorphisms between  $C^*$ -algebras are necessarily  $*$ -preserving. Hence, we shall refer to (weak) property  $(T_{\mathcal{H}})$  simply as (weak) property (T).

**Warning!** Our terminology differs from that of Bekka and Ng for  $C^*$ -algebras: While Bekka and Ng use the terms *strong property* (T) and *property* (T) for the stronger, respectively, weaker version, we prefer the terms *property* (T) and *weak property* (T). In particular, *property* (T) is the weaker version for them and the stronger version for us. As we shall see in a moment, our terminology is better aligned with the terminology on the group level. Moreover, we avoid confusion with the established notion of strong property (T) for groups.

*Remark II.3.5.* In the cases where  $E^{\mathcal{A}}$  is a complemented subspace of  $E$  and  $P$  is the projection onto  $E^{\mathcal{A}}$  along its complement, we may take  $\eta_i = P\xi_i$  in the definition of property  $(T_{\mathcal{E}})$ .

In his definition of property (T) for a unital  $C^*$ -algebra  $\mathcal{A}$  (see Definition 6 in [Bek05]), Bekka considered  $\mathcal{A}$ -bimodules admitting a net of almost  $\mathcal{A}$ -central unit vectors rather than strictly almost  $\mathcal{A}$ -central unit vectors. When  $\mathcal{A}$  is unital, his definition and that of Ng coincides (see Proposition 2.5(b) in [Ng13]). In the more general setting of property  $(T_{\mathcal{E}})$  for Banach algebras, the same phenomenon happens.

**Proposition II.3.6.** *Let  $\mathcal{A}$  be a Banach algebra with a bounded approximate unit and let  $\mathcal{E}$  be a class of Banach spaces. Assume  $\mathcal{A}$  satisfies the following property: whenever  $E \in \mathcal{E}$  is an essential  $\mathcal{A}$ -bimodule admitting a net  $(\xi_i)_{i \in I}$  of almost  $\mathcal{A}$ -central unit vectors then  $E$  contains a net  $(\eta_i)_{i \in I}$  of central vectors such that  $\|\xi_i - \eta_i\|_E \rightarrow 0$ . Then  $\mathcal{A}$  has property  $(T_{\mathcal{E}})$ . The converse is true if  $\mathcal{A}$  is unital.*

*Proof.* Consider the canonical embedding  $\zeta : \mathcal{A} \hookrightarrow M(\mathcal{A})$ . If  $F \subset \mathcal{A}$  is a finite subset then  $\zeta(F)$  is a finite, hence strictly compact, subset of  $M(\mathcal{A})$ . Now, for any  $\xi \in E$ , we have

$$\sup_{a \in F} \|a \cdot \xi - \xi \cdot a\|_E = \sup_{x \in \zeta(F)} \|x \cdot \xi - \xi \cdot x\|_E.$$

Hence, any net of strictly almost  $\mathcal{A}$ -central vectors is automatically also a net of almost  $\mathcal{A}$ -central vectors. It is thus clear that property  $(T_{\mathcal{E}})$  for  $\mathcal{A}$  is implied by the mentioned property.

For the converse implication, assume that  $\mathcal{A}$  is unital. Let  $S$  be any strictly compact subset of  $M(\mathcal{A})$ . Because  $\mathcal{A}$  is unital, any element in  $M(\mathcal{A})$  is of the form  $(L_a, R_a)$ , for some  $a \in \mathcal{A}$ . Thus,  $S = \zeta(F)$  for some (not necessarily finite) subset  $F$  of  $\mathcal{A}$ . For a given  $\varepsilon > 0$ , we apply Lemma II.2.8(ii) with the identity of  $\mathcal{A}$  in place of  $a_0$  to obtain a finite number of elements  $a_1, \dots, a_n \in F$  such that, for any  $a \in F$ , there is a  $k \in \{1, \dots, n\}$  for which

$$\|a - a_k\|_{\mathcal{A}} = \|L_a(1_{\mathcal{A}}) - L_{a_k}(1_{\mathcal{A}})\|_{\mathcal{A}} < \varepsilon. \quad (\text{II.2})$$

Let  $E$  be an  $\mathcal{A}$ -bimodule. For  $a \in F$ , take  $k$  such that equation (II.2) holds. Then,

$$\|a \cdot \xi - \xi \cdot a\|_E \leq \|a_k \cdot \xi - \xi \cdot a_k\|_E + 2\varepsilon \|\xi\|_E,$$

for any  $\xi \in E$ . Thus, for any  $\xi \in E$  with  $\|\xi\|_E = 1$ ,

$$\sup_{x \in \zeta(F)} \|x \cdot \xi - \xi \cdot x\|_E = \sup_{a \in F} \|a \cdot \xi - \xi \cdot a\|_E \leq \sup_{k \in \{1, \dots, n\}} \|a_k \cdot \xi - \xi \cdot a_k\|_E + 2\varepsilon.$$

So any net of vectors in  $E$  which is almost  $\mathcal{A}$ -central is also strictly almost  $\mathcal{A}$ -central. Hence, if  $E$  admits a net of almost  $\mathcal{A}$ -central unit vectors, property  $(T_{\mathcal{E}})$  of  $\mathcal{A}$  will imply the existence of a net  $(\eta_i)_{i \in I}$  of central vectors such that  $\|\xi_i - \eta_i\|_E$  converges to zero.  $\square$

*Remark II.3.7.* The similar statement to that of Proposition II.3.6 but for weak property  $(T_{\mathcal{E}})$  also holds with essentially the same proof.

Before proceeding to specific cases, we record the following permanence property:

**Proposition II.3.8.** *Let  $\mathcal{A}$  be a Banach algebra with a bounded approximate unit, let  $\mathcal{B}$  be another Banach algebra and let  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  be a bounded homomorphism with dense range. If  $\mathcal{A}$  has (weak) property  $(T_{\mathcal{E}})$ , for a class of Banach spaces  $\mathcal{E}$ , then so does  $\mathcal{B}$ .*

*Proof.* Let  $E \in \mathcal{E}$  be an essential  $\mathcal{B}$ -bimodule admitting a net  $(\xi_i)_{i \in I}$  of strictly almost  $M(\mathcal{B})$ -central unit vectors. Through precomposition with  $\varphi$ ,  $E$  becomes an  $\mathcal{A}$ -bimodule, and as such, it is essential because  $\varphi$  has dense range. We check that the net  $(\xi_i)_{i \in I}$  remains almost central for the induced  $M(\mathcal{A})$ -bimodule structure. By Theorem II.2.9,  $\varphi$  extends to a strictly continuous homomorphism  $\Phi : M(\mathcal{A}) \rightarrow M(\mathcal{B})$  and the  $M(\mathcal{A})$ -bimodule structure on  $E$  induced through  $\Phi$  from the  $M(\mathcal{B})$ -bimodule structure agrees with the extension of the  $\mathcal{A}$ -bimodule structure induced through  $\varphi$  from the  $\mathcal{B}$ -bimodule structure. Let  $S \subset M(\mathcal{A})$  be

any strictly compact subset. The image of  $S$  under  $\Phi$  is then a strictly compact subset of  $M(\mathcal{B})$ . Hence,

$$\sup_{x \in S} \|x \cdot \xi_i - \xi_i \cdot x\|_E = \sup_{x \in S} \|\Phi(x) \cdot \xi_i - \xi_i \cdot \Phi(x)\|_E = \sup_{y \in \Phi(S)} \|y \cdot \xi_i - \xi_i \cdot y\|_E \rightarrow 0,$$

and so,  $(\xi_i)_{i \in I}$  is a net of strictly almost  $\mathcal{A}$ -central unit vectors. Now, if  $\xi$  is any  $\mathcal{A}$ -central vector, then density of the image of  $\mathcal{A}$  under  $\varphi$  implies that  $\xi$  must also be  $\mathcal{B}$ -central. Hence, if  $\mathcal{A}$  has (weak) property  $(T_{\mathcal{E}})$  then so does  $\mathcal{B}$ .  $\square$

### II.3.1 Locally compact groups and their pseudofunction algebras

In this section we provide a characterization of (weak) property  $(T_{\mathcal{E}})$  of a locally compact group  $G$  in terms of (weak) property  $(T_{\mathcal{E}})$  of  $F_{\mathcal{E}}^*(G)$  for a class of Banach spaces  $\mathcal{E}$ ; see Theorem II.3.10A and II.3.10B. It generalizes the similar result of Bekka and Ng in Theorem 1 in [BN19] from the  $C^*$ -algebra setting to the Banach algebra setting. The generalization comes in two versions reflecting the fact that, unlike in the Hilbert spaces setting, property  $(T_{\mathcal{E}})$  for a group need not be equivalent to its weak relative. The proof relies on a natural way of constructing an isometric representation from an  $F_{\mathcal{E}}^*(G)$ -bimodule, and vice versa.

Let  $\mathcal{E}$  be a class of Banach spaces and let  $E \in \mathcal{E}$  be an essential  $F_{\mathcal{E}}^*(G)$ -bimodule with left and right actions

$$\varphi : F_{\mathcal{E}}^*(G) \rightarrow \mathcal{B}(E) \quad \text{and} \quad \psi : F_{\mathcal{E}}^*(G)^{\text{op}} \rightarrow \mathcal{B}(E).$$

By Proposition II.2.4 and Remark II.2.11,  $\varphi$  and  $\psi$  are induced from isometric representations of  $G$ , respectively  $G^{\text{op}}$ , which we shall also denote by  $\varphi$  and  $\psi$ . Since  $\varphi$  and  $\psi$  have commuting ranges as left and right actions of  $F_{\mathcal{E}}^*(G)$ , their ranges as group representations commute as well. We construct a new isometric representation  $\pi$  of  $G$  on  $E$  by setting

$$\pi(t)\xi = \varphi(t)\psi(t^{-1})\xi, \quad \text{for all } t \in G \text{ and } \xi \in E. \quad (\text{II.3})$$

It is easy to see that a vector  $\xi \in E$  is  $G$ -invariant if and only if it is  $F_{\mathcal{E}}^*(G)$ -central. Further, for every vector  $\xi \in E$  and every  $t \in G$ , Proposition II.2.10 yields that

$$\|\pi(t)\xi - \xi\|_E = \|\varphi(t)\xi - \psi(t)\xi\|_E = \|(L_t, \Delta(t^{-1})R_{t^{-1}}) \cdot \xi - \xi \cdot (L_t, \Delta(t^{-1})R_{t^{-1}})\|_E,$$

Because the embedding  $G \hookrightarrow M(F_{\mathcal{E}}^*(G))$  is continuous when  $M(F_{\mathcal{E}}^*(G))$  is equipped with the strict topology, we see that any net of strictly almost  $M(F_{\mathcal{E}}^*(G))$ -central vectors is also a net of almost  $G$ -invariant vectors.

Now let  $(\pi, E)$  be an isometric representation of  $G$  with  $E \in \mathcal{E}$ . Then  $\pi$  induces a non-degenerate, contractive representation of  $F_{\mathcal{E}}^*(G)$  on  $E$ . Further, since the trivial representation  $1_G$  belongs to  $\text{Rep}_{\mathcal{E}}(G)$ , it extends to  $F_{\mathcal{E}}^*(G)$ . This induces an essential  $F_{\mathcal{E}}^*(G)$ -bimodule structure on  $E$  with left action  $\pi$  and right action

$1_G$ . It is easy to see that the  $F_{\mathcal{E}}^*(G)$ -central vectors for this bimodule structure are exactly the  $G$ -invariant vectors. Further, for each  $\xi \in E$  and each  $f \in C_c(G)$ ,

$$\begin{aligned} \|\pi(f)\xi - 1_G(f)\xi\|_E &= \left\| \int_G f(s) (\pi(s)\xi - \xi) \, d\mu_G(s) \right\|_E \\ &\leq \int_G |f(s)| \|\pi(s)\xi - \xi\|_E \, d\mu_G(s) \\ &\leq \sup_{s \in \text{supp } f} \|\pi(s)\xi - \xi\|_E \|f\|_1. \end{aligned}$$

Let  $x \in F_{\mathcal{E}}^*(G)$ . For any  $\varepsilon > 0$ , we can find  $f \in C_c(G)$  such that  $\|x - f\|_{F_{\mathcal{E}}^*(G)} < \varepsilon$ . Then

$$\|\pi(x)\xi - 1_G(x)\xi\|_E < \|\pi(f)\xi - 1_G(f)\xi\|_E + 2\varepsilon.$$

Hence, if  $(\xi_i)_{i \in I}$  is a net in  $E$  of almost invariant unit vectors then it is almost  $F_{\mathcal{E}}^*(G)$ -central for the bimodule structure on  $E$  with left action  $\pi$  and right action  $1_G$ . In fact, as we shall see next, it will be almost central for the extension of the bimodule structure to the multiplier algebra. We show this in the following technical lemma, which is based on the proof of Proposition 10 in [BN19].

**Lemma II.3.9.** *Let  $(\pi, E)$  be an isometric representation of the locally compact group  $G$  with  $E$  in the class  $\mathcal{E}$ , and view  $E$  as an  $F_{\mathcal{E}}^*(G)$ -bimodule with left action  $\pi$  and right action  $1_G$ . Then any net of almost  $F_{\mathcal{E}}^*(G)$ -central unit vectors is automatically strictly almost  $F_{\mathcal{E}}^*(G)$ -central.*

*Proof.* Suppose  $(\xi_i)_{i \in I}$  is a net of almost  $F_{\mathcal{E}}^*(G)$ -central unit vectors in  $E$ . Fix  $a_0 \in F_{\mathcal{E}}^*(G)$  such that  $1_G(a_0) = 1$ . Then

$$\|\|\pi(a_0)\xi_i\|_E - 1\| \leq \|\pi(a_0)\xi_i - 1_G(a_0)\xi_i\|_E,$$

for all  $i \in I$ , and so,

$$\lim_i \|\pi(a_0)\xi_i\|_E = 1.$$

We may assume that  $\pi(a_0)\xi_i$  is non-zero for all  $i \in I$  as we can otherwise pass to a subnet. Define, for each  $i \in I$ ,

$$\eta_i = \frac{\pi(a_0)\xi_i}{\|\pi(a_0)\xi_i\|_E}.$$

We claim that  $(\eta_i)_{i \in I}$  constitutes a net of strictly almost  $F_{\mathcal{E}}^*(G)$ -central unit vectors. To see this, let  $S$  be any strictly compact subset of  $M(F_{\mathcal{E}}^*(G))$ . Given  $\varepsilon > 0$ , we can find a finite collection of elements  $x_1, \dots, x_n$  of  $S$  such that, for every  $x \in S$ , there is a  $k \in \{1, \dots, n\}$  for which

$$\|xa_0 - x_k a_0\|_{F_{\mathcal{E}}^*(G)} < \frac{\varepsilon}{12}. \quad (\text{II.4})$$

Take  $i_0 \in I$  such that the following hold, for all  $i \succ i_0$  and all  $k = 1, \dots, n$ ,

$$\begin{aligned} \|\pi(a_0)\xi_i\|_E &\geq \frac{1}{2}, \\ \|\pi(a_0)\xi_i - 1_G(a_0)\xi_i\|_E &< \frac{\varepsilon}{4 \sup_{y \in S} \|y\|}, \\ \|\pi(x_k a_0)\xi_i - 1_G(x_k a_0)\xi_i\|_E &< \frac{\varepsilon}{12}. \end{aligned}$$

Now, given  $x \in S$ , take  $k \in \{1, \dots, n\}$  such that (II.4) holds. Then

$$\begin{aligned} \|x \cdot \eta_i - \eta_i \cdot x\|_E &= \frac{1}{\|\pi(a_0)\xi_i\|_E} \|\pi(x)\pi(a_0)\xi_i - 1_G(x)\pi(a_0)\xi_i\|_E \\ &\leq 2\|\pi(xa_0)\xi_i - \pi(x_k a_0)\xi_i\|_E + 2\|\pi(x_k a_0)\xi_i - 1_G(x_k a_0)\xi_i\|_E \\ &\quad + 2\|1_G(x_k a_0)\xi_i - 1_G(x a_0)\xi_i\|_E + 2\|1_G(x a_0)\xi_i - 1_G(x)\pi(a_0)\xi_i\|_E \\ &\leq \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + |1_G(x)| \frac{\varepsilon}{2 \sup_{y \in S} \|y\|} < \varepsilon. \end{aligned}$$

Thus,  $(\eta_i)_{i \in I}$  is indeed a net of strictly almost  $F_{\mathcal{E}}^*(G)$ -central unit vectors in  $E$ . Now, by construction of the net  $(\eta_i)_{i \in I}$ , the norm difference  $\|\xi_i - \eta_i\|_E$  converges to zero. Hence,  $(\xi_i)_{i \in I}$  is a net of strictly almost  $F_{\mathcal{E}}^*(G)$ -central unit vectors in  $E$ , as well.  $\square$

Theorem II.3.10A below, which is one of our main results, relates property  $(T_{\mathcal{E}})$  for a locally compact group  $G$  with property  $(T_{\mathcal{E}})$  of its associated symmetrised  $\mathcal{E}$ -pseudofunction algebra.

**Theorem II.3.10A.** *Let  $G$  be a locally compact group and let  $\mathcal{E}$  be a class of Banach spaces. The following are equivalent:*

- (i)  $G$  has property  $(T_{\mathcal{E}})$ ,
- (ii)  $F_{\mathcal{E}}^*(G)$  has property  $(T_{\mathcal{E}})$ .

*Proof.* (i)  $\Rightarrow$  (ii): Assume that  $G$  has property  $(T_{\mathcal{E}})$  and let  $E \in \mathcal{E}$  be an essential  $F_{\mathcal{E}}^*(G)$ -bimodule admitting a net  $(\xi_i)_{i \in I}$  of strictly almost  $F_{\mathcal{E}}^*(G)$ -central unit vectors. Then  $(\xi_i)_{i \in I}$  is almost  $G$ -invariant for the isometric representation  $\pi$  of  $G$  induced by the  $F_{\mathcal{E}}^*(G)$ -bimodule structure. By the assumption that  $G$  has property  $(T_{\mathcal{E}})$ , we obtain a net of  $G$ -invariant vectors  $(\eta_i)_{i \in I}$  such that  $\|\xi_i - \eta_i\|_E \rightarrow 0$ . Since the  $C_c(G)$  is dense in  $F_{\mathcal{E}}^*(G)$ , we see that each  $\eta_i$  is  $F_{\mathcal{E}}^*(G)$ -central. Thus,  $F_{\mathcal{E}}^*(G)$  has property  $(T_{\mathcal{E}})$ .

(ii)  $\Rightarrow$  (i): Assume  $F_{\mathcal{E}}^*(G)$  has property  $(T_{\mathcal{E}})$  and let  $(\pi, E)$  be an isometric representation of  $G$  on a Banach space  $E$  in  $\mathcal{E}$  admitting a net  $(\xi_i)_{i \in I}$  of almost invariant unit vectors. Then  $(\xi_i)_{i \in I}$  is almost  $F_{\mathcal{E}}^*(G)$ -central for the bimodule structure on  $E$  with left action  $\pi$  and right action  $1_G$ . Hence,  $(\xi_i)_{i \in I}$  is automatically strictly almost  $F_{\mathcal{E}}^*(G)$ -central, by Lemma II.3.9. By the assumption that  $F_{\mathcal{E}}^*(G)$  has property  $(T_{\mathcal{E}})$ , we obtain a net  $(\eta_i)_{i \in I}$  of  $F_{\mathcal{E}}^*(G)$ -central vectors such that  $\|\xi_i - \eta_i\|_E \rightarrow 0$ . Hence,  $G$  has property  $(T_{\mathcal{E}})$ .  $\square$

*Remark II.3.11.* Theorem II.3.10A also holds with  $F_{\mathcal{E}}(G)$  in place of  $F_{\mathcal{E}}^*(G)$ . In fact, the proof shows the following stronger statement: If  $G$  has property  $(T_{\mathcal{E}})$  then  $F_{\mathcal{R}}(G)$  has property  $(T_{\mathcal{E}})$  for any class  $\mathcal{R}$  of isometric Banach space representations of  $G$ . Further, a sufficient condition for the converse implication is that  $\mathcal{R}$  contains the class of all isometric representations of  $G$  on a space in  $\mathcal{E}$ .

The similar statement holds when exchanging property  $(T_{\mathcal{E}})$  with weak property  $(T_{\mathcal{E}})$ . The proof is the same mutatis mutandis, and so, we omit it.

**Theorem II.3.10B.** *Let  $G$  be a locally compact group and  $\mathcal{E}$  a class of Banach spaces. The following are equivalent:*

- (i)  $G$  has weak property  $(T_{\mathcal{E}})$ ,
- (ii)  $F_{\mathcal{E}}^*(G)$  has weak property  $(T_{\mathcal{E}})$ .

*Remark II.3.12.* When  $\mathcal{E}$  is a class satisfying either of the two conditions of Proposition II.2.19 so that property  $(T_{\mathcal{E}})$  and weak property  $(T_{\mathcal{E}})$  for the group are equivalent, the two theorems II.3.10A and II.3.10B can be merged into one. This holds, in particular, when  $\mathcal{E}$  is the class of complex Hilbert spaces, in which case we recover the similar result of Bekka and Ng in Theorem 1 in [BN19].

As an immediate corollary to Theorem II.3.10A and to Theorem A in [BFGM07], we obtain the following equivalence:

**Corollary II.3.13.** *Let  $G$  be a locally compact group and let  $1 \leq p, q < \infty$ . Then  $F_{L^p}^*(G)$  has property  $(T_{L^p})$  if and only if  $F_{L^q}^*(G)$  has property  $(T_{L^q})$ .*

### II.3.2 Property $(T_{L^q})$ for $F_{L^p}^*(G)$

Let  $\mathcal{E}$  be the class of  $L^p$ -spaces. One may view the associated Banach \*-algebras  $F_{L^p}^*(G)$  as interpolating between  $L^1(G)$  and the universal group  $C^*$ -algebra  $C^*(G)$  as  $p$  varies from 1 to 2. Precisely, for  $1 \leq q \leq p \leq 2$ , the identity on  $L^1(G)$  extends to a contractive homomorphism  $F_{L^q}^*(G) \rightarrow F_{L^p}^*(G)$ .<sup>1</sup> With this in mind, we obtain the following results as consequences to Theorem II.3.10A:

**Corollary II.3.14.** *Let  $G$  be a second countable locally compact group with property  $(T)$  and let  $1 \leq p \leq 2$ . Then  $F_{L^p}^*(G)$  has property  $(T_{L^q})$ , for all  $1 \leq q \leq p$  and all  $p' \leq q < \infty$ , where  $p'$  is the Hölder conjugate of  $p$ .*

*Proof.* Since  $G$  has property  $(T)$ , it has property  $(T_{L^q})$ , for every  $1 \leq q < \infty$ , by Theorem A(i) in [BFGM07], and so,  $F_{L^q}^*(G)$  has property  $(T_{L^q})$ , by Theorem II.3.10A. For  $1 \leq q \leq p$  or  $p' \leq q < \infty$ , the identity on  $L^1(G)$  extends to a contractive homomorphism  $F_{L^q}^*(G) \rightarrow F_{L^p}^*(G)$  with dense range. It follows by Proposition II.3.8 that  $F_{L^p}^*(G)$  has property  $(T_{L^q})$ .  $\square$

For discrete groups we obtain a similar result also for parameters  $q$  in the interval between  $p$  and  $p'$ .

**Corollary II.3.15.** *Let  $\Gamma$  be a discrete group with property  $(T)$  and let  $1 \leq p \leq 2$ . Then  $F_{L^p}^*(\Gamma)$  has property  $(T_{L^q})$ , for all  $1 \leq q < \infty$ .*

<sup>1</sup>This follows from the similar result in the classical (non-symmetrized) setting by Gardella and Thiel (see Theorem 2.30 in [GT14]), but it can also be proven more directly in the symmetrized setting via interpolation theory. The latter proof is part of work in progress of the first named author.

*Proof.* It suffices to show the statement for  $p < q < p'$ ; the cases where  $q$  is in between 1 and  $p$  or greater than  $p'$  are covered in Corollary II.3.14. As in the proof of Corollary II.3.14, it follows from Theorem A(i) [BFGM07] and Theorem II.3.10A that  $F_{L^q}^*(\Gamma)$  has property  $(T_{L^q})$ . Let  $L^q(\Omega, \nu)$  be an  $F_{L^p}^*(\Gamma)$ -bimodule admitting a net  $(\xi_i)_{i \in I}$  of almost  $F_{L^p}^*(\Gamma)$ -central unit vectors. By construction of  $F_{L^q}^*(\Gamma)$ , we see that the  $F_{L^p}^*(\Gamma)$ -bimodule actions extend continuously to  $F_{L^q}^*(\Gamma)$ . Let  $\{x_1, \dots, x_n\}$  be any finite subset of  $F_{L^q}^*(\Gamma)$  and let  $\varepsilon > 0$ . Take  $f_1, \dots, f_n \in \ell^1(\Gamma)$  such that  $\|x_j - f_j\|_{F_{L^q}^*(\Gamma)} < \varepsilon$ , for  $j = 1, \dots, n$ . Then,

$$\sup_{j \in \{1, \dots, n\}} \|x_j \cdot \xi_i - \xi_i \cdot x_j\|_{L^q(\Omega, \nu)} < \sup_{j \in \{1, \dots, n\}} \|f_j \cdot \xi_i - \xi_i \cdot f_j\|_{L^q(\Omega, \nu)} + 2\varepsilon.$$

Because we can view  $\{f_1, \dots, f_n\}$  as a finite subset of  $F_{L^p}^*(\Gamma)$ , the supremum on the right-hand side can be made arbitrarily small when  $i$  is chosen large enough. It follows that  $(\xi_i)_{i \in I}$  is almost central for the  $F_{L^q}^*(\Gamma)$ -bimodule structure. By Proposition II.3.6, property  $(T_{L^q})$  for  $F_{L^q}^*(\Gamma)$  then implies the existence of a net  $(\eta_i)_{i \in I}$  in  $L^q(\Omega, \nu)$  consisting of  $F_{L^q}^*(\Gamma)$ -central vectors such that  $\|\xi_i - \eta_i\|_{L^q(\Omega, \nu)}$  converges to zero. As the  $F_{L^p}^*(\Gamma)$ -bimodule actions are the precomposition of the  $F_{L^q}^*(\Gamma)$ -bimodule actions with the canonical contractive homomorphism  $F_{L^p}^*(\Gamma) \rightarrow F_{L^q}^*(\Gamma)$ , the net  $(\eta_i)_{i \in I}$  is also central for the  $F_{L^p}^*(\Gamma)$ -bimodule structure. We conclude from Proposition II.3.6 that  $F_{L^p}^*(\Gamma)$  has property  $(T_{L^q})$ .  $\square$

## II.4 Property $(T_{L^p})$ for symmetrized $p$ -pseudofunction algebras

In this section, we continue to focus our attention to the class of  $L^p$ -spaces on  $\sigma$ -finite measure spaces, where  $1 \leq p < \infty$ . Due to Theorem II.3.10A,  $G$  has property  $(T_{L^p})$  if and only if  $F_{L^p}^*(G)$  has property  $(T_{L^p})$ . For  $p = 2$ , we recover the result of Bekka and Ng in Theorem 1 in [BN19] that  $G$  has property (T) if and only if the universal group  $C^*$ -algebra  $C^*(G)$  of  $G$  has property (T). After establishing their result, Bekka and Ng ask if  $C^*(G)$  can be replaced by the reduced group  $C^*$ -algebra  $C_r^*(G)$  of  $G$ . In this section, we ask the same question but in the more general setting of actions on  $L^p$ -spaces. Here, the role of the reduced group  $C^*$ -algebra is played by  $F_{\lambda_p}^*(G)$ . Thus, we ask if (or when) property  $(T_{L^p})$  for the group is captured by  $F_{\lambda_p}^*(G)$ . We shall see that this is the case when  $G$  is a discrete group (see Theorem II.4.4). This result generalizes that of Bekka and Ng in the case of discrete groups. Our proof relies on a generalisation of Fell's absorption principle to isometric representations on  $L^p$ -spaces. Further, when  $G$  is discrete, we show that weak property  $(T_{SL^p})$  for  $F_{\lambda_p}^*(G)$  implies property  $(T_{L^p})$  for the group (see Theorem II.4.12), where  $SL^p$  is the class of closed subspaces of  $L^p$ -spaces on  $\sigma$ -finite measure spaces.

Let  $(\pi, L^p(\Omega, \nu))$  be an isometric  $L^p$ -representation of the locally compact group  $G$ . We denote by  $\text{id}$  the trivial representation of  $G$  on  $L^p(\Omega, \nu)$  and by  $\lambda_p$  the left-regular representation of  $G$  on  $L^p(G)$ . Consider the  $L^p$ -space

$$E = L^p(G, L^p(\Omega, \nu)). \tag{II.5}$$

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This space contains the algebraic tensor product  $L^p(G) \odot L^p(\Omega, \nu)$  as a dense subspace. Hence,  $\lambda_p \otimes \pi$  and  $\lambda_p \otimes \text{id}$  define isometric representations of  $G$  on  $E$ . For  $p = 2$ , we know from Fell's absorption principle that these two representations are unitarily equivalent. For general  $1 \leq p < \infty$ , an  $L^p$ -version of the Fell's absorption principle was shown in Proposition 5.1 [Run04]. We state it in Proposition II.4.1 below in the form we need it.

**Proposition II.4.1** ( $L^p$ -version of Fell's absorption principle). *Let  $G$  be a locally compact group, let  $1 \leq p < \infty$ , and let  $(\pi, L^p(\Omega, \nu))$  be an isometric representation of  $G$ . Then  $\lambda_p \otimes \pi$  and  $\lambda_p \otimes \text{id}$  are equivalent in the sense that they are intertwined by a surjective isometry of  $L^p(G; L^p(\Omega, \nu))$ .*

*Remark II.4.2.* In Proposition II.4.1, one can exchange the left regular representation with the right regular representation,  $\rho_p$ . That is,  $\rho_p \otimes \pi$  is equivalent to  $\rho_p \otimes \text{id}$ . The proof is the same mutatis mutandis.

Using the  $L^p$ -version of Fell's absorption principle, we can construct an  $F_{\lambda_p}^*(G)$ -bimodule on  $E$  from the isometric representation  $(\pi, L^p(\Omega, \nu))$  of  $G$  as follows: Set  $\varphi = \lambda_p \otimes \text{id}$  and  $\psi = \rho_p \otimes \pi$  of  $G$  on  $E$ . Clearly,  $\varphi$  integrates to a representation of  $F_{\lambda_p}^*(G)$ , and  $\psi$  does as well by Proposition II.4.1 and the remark following it. Thus,  $E$  is an  $F_{\lambda_p}^*(G)$ -bimodule with left action  $\varphi$  and right action  $\psi^{\text{op}} = \psi \circ \tilde{\square}$ .

**Lemma II.4.3.** *Let  $G$  be a locally compact group, let  $(\pi, L^p(\Omega, \nu))$  be an isometric representation and let  $E$  be the  $F_{\lambda_p}^*(G)$ -bimodule from equation (II.5). If  $\eta \in E$  is central then*

$$\pi(s)\eta(t) = \eta(sts^{-1}),$$

for all  $s \in G$  and  $\mu_G$ -almost all  $t \in G$ .

*Proof.* Let  $\eta \in E$  be central. Then  $\eta$  is also central for the extension of the bimodule structure to  $M(F_{\lambda_p}^*(G))$ . Hence, for every  $s \in G$ ,  $\psi(s)\eta = \psi^{\text{op}}(s^{-1})\eta = \varphi(s^{-1})\eta$ , where the equality is in  $E$ . It follows that

$$\pi(s)\eta(t) = \psi(s)\eta(ts^{-1}) = \varphi(s^{-1})\eta(ts^{-1}) = \eta(sts^{-1}),$$

for each  $s \in G$  and for  $\mu_G$ -almost every  $t \in G$ . □

### II.4.1 Property $(T_{L^p})$ for $F_{\lambda_p}^*$ for discrete groups

We show next that property  $(T_{L^p})$  for a discrete group  $\Gamma$  is detected by its (symmetrized)  $p$ -pseudofunction algebra. Recall that pseudofunction algebras of discrete groups are unital, and we can therefore use the equivalent definition of property  $(T_{L^p})$  from Proposition II.3.6.

**Theorem II.4.4.** *Let  $\Gamma$  be a discrete group. For each  $1 \leq p < \infty$ , the following are equivalent:*

- (i)  $\Gamma$  has property  $(T_{L^p})$ ,
- (ii)  $F_{L^p}^*(\Gamma)$  has property  $(T_{L^p})$ ,

(iii)  $F_{\lambda_p}^*(\Gamma)$  has property  $(T_{L^p})$ .

Our proof uses the ideas of the proof of Theorem 9 in [BN19].

*Proof.* (i)  $\Rightarrow$  (ii) is covered by Theorem II.3.10A and (ii)  $\Rightarrow$  (iii) follows from Proposition II.3.8. It remains to show (iii)  $\Rightarrow$  (i). Suppose  $(\pi, L^p(\Omega, \nu))$  is an isometric representation of  $\Gamma$  and let  $E$  be the  $F_{\lambda_p}^*(\Gamma)$ -bimodule from equation (II.5). Given  $\xi \in L^p(\Omega, \nu)$  set  $\zeta = \delta_e \otimes \xi \in E$ . For each  $f \in C_c(\Gamma)$ , we have

$$\begin{aligned} f \cdot \zeta &= \varphi(f)(\zeta) = \sum_{r \in \Gamma} f(r) \delta_r \otimes \xi, \\ \zeta \cdot f &= \psi^{\text{op}}(f)(\zeta) = \sum_{r \in \Gamma} f(r) \delta_r \otimes \pi(r^{-1})\xi. \end{aligned}$$

We compute

$$\begin{aligned} \|f \cdot \zeta - \zeta \cdot f\|_E &= \left\| \sum_{r \in \Gamma} f(r) \delta_r \otimes (\xi - \pi(r^{-1})\xi) \right\|_E \\ &= \left( \sum_{s \in \Gamma} \|f(s)(\xi - \pi(s^{-1})\xi)\|_{L^p(\Omega, \nu)}^p \right)^{1/p} \\ &\leq \|f\|_1 \sup_{t \in \text{supp}(f)} \|\pi(t)\xi - \xi\|_{L^p(\Omega, \nu)}. \end{aligned}$$

Suppose  $\pi$  admits a net  $(\xi_i)_{i \in I}$  of almost  $\Gamma$ -invariant unit vectors in  $L^p(\Omega, \nu)$ . For each  $i \in I$ , set  $\zeta_i = \delta_e \otimes \xi_i$ . Given  $f \in C_c(\Gamma)$  and  $\varepsilon > 0$ , pick  $i_{f, \varepsilon} \in I$  such that, for all  $i \succ i_{f, \varepsilon}$ ,

$$\sup_{t \in \text{supp}(f)} \|\pi(t)\xi_i - \xi_i\|_{L^p(\Omega, \nu)} < \varepsilon / \|f\|_1.$$

By the above calculations, we see that  $\|f \cdot \zeta_i - \zeta_i \cdot f\|_E < \varepsilon$ , for all  $i \succ i_{f, \varepsilon}$ . Since  $C_c(\Gamma)$  is dense in  $F_{\lambda_p}^*(\Gamma)$ , we deduce that  $(\zeta_i)_{i \in I}$  is an almost  $F_{\lambda_p}^*(\Gamma)$ -central net. By the assumption that  $F_{\lambda_p}^*(\Gamma)$  has property  $(T_{L^p})$ , we obtain a net  $(\eta_i)_{i \in I}$  of  $F_{\lambda_p}^*(\Gamma)$ -central vectors satisfying

$$\|\delta_e \otimes \xi_i - \eta_i\|_{L^p(G; L^p(\Omega, \nu))} \rightarrow 0,$$

By Lemma II.4.3, we have equality  $\pi(s)(\eta_i(t)) = \eta_i(s^{-1}ts)$ , for all  $s, t \in G$  and every  $i \in I$ . Hence,  $\eta_i(e)$  is a  $\Gamma$ -invariant vector in  $L^p(\Omega, \nu)$ . Further,

$$\|\xi_i - \eta_i(e)\|_{L^p(\Omega, \nu)} \leq \|\delta_e \otimes \xi_i - \eta_i\|_E \rightarrow 0.$$

Hence, we conclude that  $\Gamma$  has property  $(T_{L^p})$ .  $\square$

*Remark II.4.5.* Theorem II.4.4 also holds with  $F_{L^p}(\Gamma)$  and  $F_{\lambda_p}(\Gamma)$  in place of  $F_{L^p}^*(\Gamma)$  and  $F_{\lambda_p}^*(\Gamma)$ . The proof is the same.

*Remark II.4.6.* The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) in Theorem II.4.4 hold for general second countable locally compact groups. For  $p = 2$ , it is shown in Example 6 in [BN19] that the implication (iii) $\Rightarrow$ (i) fails for general locally compact groups, but it holds for IN groups (see Theorem 9(c) in [BN19]) – a class of locally

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compact groups which contains, but is strictly larger than, the class of discrete groups. When  $p \neq 2$  and  $G$  is a locally compact non-discrete IN group, the proof given above with the appropriate adjustments (analogous to [BN19]) shows that property  $(T_{L^p})$  for  $F_{\lambda_p}^*(G)$  implies weak property  $(T_{L^p})$  for  $G$ . For countable discrete groups the equivalence between weak property  $(T_{L^p})$  and property  $(T_{L^p})$  is known (see Theorem C in [Elk24]), however it is open whether this is true for general locally compact groups.

As a corollary to Theorem II.4.4, we establish a relation between property  $(T_{L^p})$  and amenability on the level of the  $p$ -pseudofunction algebra. The first part of the proof is analogous to the proof given in [GT14] of their Theorem 3.11.

**Corollary II.4.7.** *Let  $\Gamma$  be a discrete group. Suppose  $F_{\lambda_p}^*(\Gamma)$  has property  $(T_{L^p})$  and it is amenable as a Banach algebra. Then it is finite dimensional.*

*Proof.* By Theorem 2.3.1 in [Run20], amenability of  $F_{\lambda_p}^*(\Gamma)$  implies that of the reduced group  $C^*$ -algebra  $C_r^*(\Gamma)$ , which in turn yields amenability of  $\Gamma$ , by Theorem 2.6.8 in [BO08]. Moreover, Theorem II.4.4 combined with Remark II.2.16 imply that  $\Gamma$  has Kazhdan property (T). It is well-known that amenability and property (T) of discrete groups imply finiteness. Therefore  $F_{\lambda_p}^*(\Gamma) = \mathbb{C}\Gamma$  is finite dimensional.  $\square$

### II.4.2 Weak property $(T_{SL^p})$ for discrete groups

Denote by  $SL^p$  the class of closed subspaces of  $L^p$ -spaces on  $\sigma$ -finite measure spaces, for some fixed  $1 \leq p < \infty$ . We show that, for a discrete group  $\Gamma$ , weak property  $(T_{SL^p})$  implies property  $(T_{L^p})$ . Moreover, we show that weak property  $(T_{SL^p})$  for  $F_{\lambda_p}^*(\Gamma)$  is intermediate to the two by adapting the proof of Bekka in [Bek05] that property (T) for a discrete group  $\Gamma$  is implied by property (T) for  $C_r^*(\Gamma)$ . Along the way, we show that property  $(T_{L^p})$  for  $\Gamma$  is implied by the property that isometric representations on spaces in  $SL^p$  with almost  $\Gamma$ -invariant vectors necessarily must have a finite dimensional subrepresentation. This should be compared with the similar well-known characterization of property (T) in the setting of unitary representations (see Theorem 1 in [BV93]). The original proof of in the setting of unitary representations via the characterization of Kazhdan's property (T) of Delorme and Guichardet utilizes Schönberg's theorem and a GNS-construction to construct a unitary representation with almost invariant vectors. Since we are concerned with isometric representations on  $L^p$ -spaces, this route does not seem feasible to us. In order to circumvent this, we will provide an alternative proof based on notions from ergodic theory, which employs an idea used in [BFGM07].

**Definition II.4.8.** Let  $\Gamma$  be a discrete group and let  $1 \leq p < \infty$ . An isometric representation of  $\Gamma$ ,  $(\pi, L^p(\Omega, \nu))$ , is called *weakly mixing* if, for each pair of finite sets  $E \subset L^p(\Omega, \nu)$  and  $F \subset L^p(\Omega, \nu)$  and each  $\varepsilon > 0$ , there exists  $t \in \Gamma$  such that

$$|\langle \pi(t)\xi, \eta \rangle| < \varepsilon,$$

for all  $\xi \in E$  and  $\eta \in F$ .

A *probability measure preserving action* (in short, a p.m.p. action) of a discrete group  $\Gamma$  on a probability space  $(\Omega, \nu)$  is a group homomorphism from  $\Gamma$  to the group of bi-measurable transformations of  $\Omega$  such that  $\nu(t.A) = \nu(A)$ , for all  $t \in \Gamma$  and all  $A \subset \Omega$  measurable.

**Definition II.4.9.** Let  $\Gamma$  be an infinite discrete group and let  $\Gamma \curvearrowright (\Omega, \mathcal{B}, \nu)$  be a p.m.p. action of  $\Gamma$  on a probability space  $(\Omega, \mathcal{B}, \nu)$ . This action is called *weakly mixing* if, for all  $\mathcal{F} \subset \mathcal{B}$  finite, we have

$$\liminf_{g \rightarrow \infty} \sum_{A, B \in \mathcal{F}} |\nu(A \cap gB) - \nu(A)\nu(B)| = 0.$$

For the sake of completeness we include the proof of the following statements.

**Proposition II.4.10.** Let  $\Gamma \curvearrowright (\Omega, \mathcal{B}, \nu)$  be a p.m.p action of the discrete group  $\Gamma$  and fix  $1 \leq p < \infty$ . If the action is weakly mixing then the Koopman representation  $\pi_0: \Gamma \rightarrow \text{Iso}(L_0^p(\Omega, \mathcal{B}, \nu))$  is weakly mixing.

*Proof.* Let  $E$  and  $F$  be finite sets of simple functions  $\sum c_{A_i} \chi_{A_i}$  given by a finite set  $\{A_i \in \mathcal{B} \mid 1 \leq i \leq n\}$  and a finite set of coefficients  $\{c_{A_i} \mid 1 \leq i \leq n\}$ , such that  $\sum c_{A_i} \nu(A_i) = 0$ . Let  $\mathcal{B}_0 \subset \mathcal{B}$  to be the set of all these finitely many measurable sets. Since the action is weakly mixing, there is a sequence  $(t_n)_n$  such that for any pair  $A, B \in \mathcal{B}_0$  we have

$$\nu(t_n A \cap B) \rightarrow \nu(A)\nu(B).$$

By bilinearity of the scalar product, we have, for any pair of simple functions in  $E$  and  $F$ ,

$$\begin{aligned} \left\langle \pi_0(t_n) \sum_i c_{A_i} \chi_{A_i}, \sum_j d_{B_j} \chi_{B_j} \right\rangle &= \sum_{i,j} c_{A_i} d_{B_j} \langle \chi_{t_n A_i}, \chi_{B_j} \rangle \\ &\rightarrow \sum_{i,j} c_{A_i} d_{B_j} \nu(A_i) \nu(B_j) = 0. \end{aligned}$$

Because simple functions are dense in  $L_0^p(\Omega, \mathcal{B}, \nu)$  and  $L_0^{p'}(\Omega, \mathcal{B}, \nu)$ , this shows that the Koopman representation is weakly mixing.  $\square$

**Lemma II.4.11.** Let  $\pi$  be an isometric representation of  $\Gamma$  on a Banach space  $X$ . If  $\pi$  is weakly mixing, then it does not have any non-zero finite dimensional subrepresentations.

*Proof.* Let  $V \subset X$  be a finite dimensional invariant subspace with basis  $\{\xi_1, \dots, \xi_d\}$ . Because  $\pi$  is weakly mixing there exists a sequence  $(t_n)_n$  in  $\Gamma$  such that  $\langle \pi(t_n) \xi_i, \eta \rangle \rightarrow 0$ , for all  $i \in \{1, \dots, d\}$  and all  $\eta \in V^*$ . Since  $V$  is finite dimensional, we may pass to a subsequence of  $(t_n)_n$  and assume that there is a  $T \in \text{Iso}(V)$  such that  $\pi(t_n) \xi \rightarrow T \xi$ , for all  $\xi \in V$ . Fix  $\xi \in V$ . For each  $\eta \in V^*$ , we see that

$$\langle T \xi, \eta \rangle = \langle \lim \pi(t_n) \xi, \eta \rangle = 0$$

Thus,  $T \xi = 0$ . As  $T$  is an isometry, this implies  $\xi = 0$ , and so,  $V$  must be zero.  $\square$

#### II.4. Property $(T_{L^p})$ for symmetrized $p$ -pseudofunction algebras

Connes and Weiss provide in [CW80] a dynamical characterization of property (T) in terms of ergodic p.m.p. actions which we recall here: A discrete group  $\Gamma$  has Kazhdan's property (T) if and only if every p.m.p. ergodic (even weakly mixing) action of  $\Gamma$  is strongly ergodic. (See also Theorem 6.3.4 in [BLV08]).

We are now ready to prove the main result of this subsection.

**Theorem II.4.12.** *Let  $\Gamma$  be a discrete group, and let  $1 \leq p < \infty$ . Each of the following implies the next:*

- (i)  $\Gamma$  has weak property  $(T_{SL^p})$ ,
- (ii)  $F_{\lambda_p}^*(\Gamma)$  has weak property  $(T_{SL^p})$ ,
- (iii) If an isometric representation  $\pi$  of  $\Gamma$  on a closed subspace of some  $L^p(\Omega, \nu)$  contains almost  $\Gamma$ -invariant vectors, then it has a finite dimensional subrepresentation,
- (iv)  $\Gamma$  has property  $(T_{L^p})$ .

*Proof.* (i)  $\Rightarrow$  (ii) follows from Theorem II.3.10B and Proposition II.3.8.

(ii)  $\Rightarrow$  (iii): Suppose  $(\pi, X)$  is an isometric representation of  $\Gamma$  with  $X \subset L^p(\Omega, \nu)$  a closed subspace. Set

$$E = \ell^p(\Gamma) \otimes_p X \cong \ell^p(\Gamma, X) \subset \ell^p(\Gamma, L^p(\Omega, \nu)).$$

Then  $E$  is an  $F_{\lambda_p}^*(\Gamma)$ -bimodule with left action  $\varphi = \lambda_p \otimes \text{id}$  and right action  $\psi^{\text{op}} = \rho_p^{\text{op}} \otimes \pi^{\text{op}}$ . Given  $\xi \in L^p(\Omega, \nu)$  consider the vector  $\zeta = \delta_e \otimes \xi$  in  $E$ . We compute

$$\begin{aligned} \|f \cdot \zeta - \zeta \cdot f\|_E &= \|\varphi(f)(\delta_e \otimes \xi) - \psi^{\text{op}}(f)(\delta_e \otimes \xi)\|_E \\ &= \left\| \sum_{s \in \Gamma} f(s) \delta_s \otimes (\xi - \pi(s^{-1})\xi) \right\|_E \\ &= \left( \sum_{s \in \Gamma} \|f(s)(\xi - \pi(s^{-1})\xi)\|_{L^p(\Omega, \nu)}^p \right)^{1/p} \\ &\leq \|f\|_p \sup_{r \in \text{supp}(f)} \|\pi(r)\xi - \xi\|_{L^p(\Omega, \nu)}. \end{aligned}$$

Assume  $F_{\lambda_p}^*(\Gamma)$  has weak property  $(T_{SL^p})$  and suppose  $\pi$  admits a net  $(\xi_i)_{i \in I}$  of almost  $\Gamma$ -invariant unit vectors in  $X$ . For each  $i \in I$ , set  $\zeta_i = \delta_e \otimes \xi_i$ . Then  $(\zeta_i)_{i \in I}$  is a net of unit vectors in  $E$ . By our above calculations  $\|f \cdot \zeta_i - \zeta_i \cdot f\|_E \rightarrow 0$ , for every  $f \in C_c(\Gamma)$ . Because  $C_c(\Gamma)$  is dense in  $F_{\lambda_p}^*(\Gamma)$ , it follows that  $(\zeta_i)_{i \in I}$  is an almost  $F_{\lambda_p}^*(\Gamma)$ -central net. Thus,  $E$  admits a non-zero central vector  $\zeta$ .

By Lemma II.4.3, we have, for each pair  $s, t \in \Gamma$ , the equality

$$\pi(s)\zeta(r) = \zeta(sr s^{-1}). \quad (\text{II.6})$$

Take  $t_0 \in \Gamma$  such that  $\zeta(t_0) \neq 0$ , and denote by  $\text{Cl}(t_0) = \{tt_0t^{-1} \mid t \in G\}$  the conjugacy class of  $t_0$ . Since  $\pi$  is an isometric representation of  $\Gamma$  on  $L^p(\Omega, \nu)$ , we

see from equation (II.6) that  $\|\zeta(tt_0t^{-1})\|_{L^p(\Omega,\nu)} = \|\zeta(t_0)\|_{L^p(\Omega,\nu)}$ , for all  $t \in \Gamma$ . From this, we deduce that

$$\|\zeta(t_0)\|_{L^p(\Omega,\nu)}^p |\text{Cl}(t_0)| = \sum_{r \in \text{Cl}(t_0)} \|\zeta(r)\|_{L^p(\Omega,\nu)}^p \leq \sum_{r \in G} \|\zeta(r)\|_{L^p(\Omega,\nu)}^p = \|\zeta\|_p^p < \infty.$$

Hence, as  $\zeta(t_0) \neq 0$ , the set  $\text{Cl}(t_0)$  must be finite. Thus, the set  $\pi(\Gamma)\zeta(t_0)$  is finite, and so, its span is a finite dimensional invariant subspace of  $X$ .

(iii)  $\Rightarrow$  (iv): Assume that  $\Gamma$  does not have property  $(T_{L^p})$  and hence not property (T) (see Remark II.2.16). Then, by work of Connes and Weiss, there exists a p.m.p weakly mixing action on a probability space  $(\Omega, \nu)$  admitting an asymptotically invariant sequence  $(B_n)_n$  of measurable subsets of  $\Omega$  with  $\nu(B_n) = 1/2$ , for all  $n$ . Since the action is weakly mixing, the Koopman representation  $\pi_0: \Gamma \rightarrow \text{Iso}(L_0^p(\Omega, \nu))$  is weakly mixing due to Proposition II.4.10. Hence, by Lemma II.4.11,  $\pi_0$  does not have any finite dimensional subrepresentation. However,  $\xi_n = 2\chi_{B_n} - 1$  provides an almost  $\Gamma$ -invariant sequence in  $L_0^p(\Omega, \nu)$ .  $\square$

*Remark II.4.13.* One can prove the implication (i)  $\Rightarrow$  (iii) in Theorem II.4.12 without passing by weak property  $(T_{SL^p})$  for  $F_{\lambda_p}^*(G)$ . Indeed, if  $\Gamma$  has property  $(T_{SL^p})$  then any isometric representation on a closed subspace of an  $L^p$ -space with almost  $\Gamma$ -invariant vectors contains an invariant vector and hence a 1-dimensional subrepresentation.

*Remark II.4.14.* In Theorem II.4.12, the core of the proof of the implication (ii)  $\Rightarrow$  (iii) is to show that the set  $\text{Cl}(t_0)$  is finite. Observe that, to reach this conclusion, it is shown that its Haar measure is finite. The conclusion that  $\text{Cl}(t_0)$  is finite is therefore contingent on the discreteness of  $\Gamma$ . It does not seem feasible to us to extend this proof to a larger class of groups.

*Remark II.4.15.* Theorem II.4.12 above remains true when exchanging  $F_{\lambda_p}^*(\Gamma)$  with its non-symmetrized relative  $F_{\lambda_p}(\Gamma)$ .

**Acknowledgements** The second named author was supported by Knut and Alice Wallenberg foundation through grant number 31001288.

We thank the organizers of the workshop on  $C^*$ -algebras and geometry of semigroups and groups held at the University of Oslo, where the initial ideas were discussed. We are grateful to Nadia Larsen, Tim de Laat, N. Christopher Phillips, Mikael de la Salle and Alain Valette for comments on an earlier draft of this paper. The first named author thanks Nadia Larsen for many discussions and Mikael de la Salle for his insightful questions and suggestions and for discussions on subtleties in the different versions of property  $(T_{\mathcal{E}})$  for groups. Finally we are very grateful to the referee for the careful reading of our article and for very useful comments.

## Erratum

1. The discussion of the multiplier algebra of  $L^1(G)$  towards the end of subsection II.2.2 may be easier to read in the light of Wendel's theorem [Wen52], which identifies  $M(L^1(G))$  with the *measure algebra*  $M(G)$  (see

#### II.4. Property $(T_{L^p})$ for symmetrized $p$ -pseudofunction algebras

also section 1.9.13 of [Pal94] for an exposition of this identification). For a group element  $s \in G$ , the multiplier  $(L_s, \Delta(s^{-1})R_{s^{-1}})$  corresponds to the Dirac measure  $\delta_s$  under the identification given by Wendel's theorem.

2. The dual representation defined in the paragraph preceding Proposition II.2.19 need not in general be strongly continuous. As a result, a given class of isometric representations may not be contained in any class which is closed under duality. The symmetrization described in the paragraph succeeding Proposition II.2.19 is therefore not always possible. For discrete groups, this problem does not occur. Further, for the classes  $L^p$  and  $QSL^p$  with  $1 < p < \infty$ , symmetrization is possible, and also the  $p$ -pseudofunction algebra can be symmetrized.
3. For  $1 < p < \infty$ ,  $L^p$ -spaces are superreflexive. In the context of Proposition II.2.19, it is thus natural to ask if the class  $L^p$  is stable under taking complemented subspaces also when  $p \neq 2$ . By Theorem 4.1 in [BL74], a subspace of an  $L^p$ -space is isometrically isomorphic to an  $L^p$ -space if and only if it is 1-complemented, that is, the range of a contractive projection. Hence, the class  $L^p$  is stable under taking complemented subspaces if and only if every complemented subspace is 1-complemented. We suspect that this is not the case, but we have not been able to find a counterexample. However, if we restrict the class  $L^p$  to contain only  $L^p$ -spaces on  $\sigma$ -finite measure spaces, an example of a complemented subspace which is not isometrically isomorphic to an  $L^p$ -space on a  $\sigma$ -finite measure space is given for  $p \notin 2\mathbb{N}$  in Theorem E in [Elk24].
4. In Proposition II.2.19, the class  $\mathcal{R}$  should be assumed to be closed both under duality and under complex conjugation. Further, observe that an implicit requirement in the definition of what it means for a class of isometric representations to be *closed under duality* is that the duals of its representations are all strongly continuous.
5. In the definition of nets of *strictly almost  $\mathcal{A}$ -central* vectors in the beginning of section II.3, it needs to be assumed that the  $\mathcal{A}$ -bimodule  $E$  is essential, so that the left and right actions are non-degenerate. This is necessary in order for the  $\mathcal{A}$ -bimodule structure to extend to an  $M(\mathcal{A})$ -bimodule structure. Moreover, it needs to be assumed that  $\mathcal{A}$  is faithful as a bimodule over itself, so that  $M(\mathcal{A})$  can be equipped with the strict topology.
6. In the second displayed equation on page 69, it is assumed that  $\xi$  is a unit vector.
7. The left-regular representation  $\lambda_p$  on  $L^p(G)$  is a complete isometry. The representation  $\lambda_p \otimes \text{id}$  on the Banach space  $E = L^p(G; L^p(\Omega, \nu))$  defined in the paragraph succeeding equation (II.5) is therefore isometric. Further, given an isometric representation  $(\pi, L^p(\Omega, \nu))$ , because Proposition II.4.1 ensures that  $\lambda_p \otimes \pi$  and  $\lambda_p \otimes \text{id}$  are intertwined by a surjective isometry, it follows that  $\lambda_p \otimes \pi$  is also isometries.

8. Definition II.4.8 can be stated for a general isometric representation of the discrete group  $\Gamma$ . For an isometric representation  $(\pi, X)$ , we replace  $L^q(\Omega, \nu)$  in Definition II.4.8 with the Banach space dual  $X^*$  and let  $\langle \cdot, \cdot \rangle$  denote the duality pairing between  $X$  and  $X^*$ .
9. In Proposition II.4.10,  $L_0^p(\Omega, \mathcal{B}, \nu)$  is the subspace of functions with mean zero.

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## Paper II. Property (T) for Banach algebras

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# Paper III

## Weak property $(T_{L^p})$ for discrete groups

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Accepted for publication in *Studia Mathematica*.

**Abstract** We show that, for a countable discrete group  $\Gamma$ , property  $(T_{L^p})$  of Bader, Furman, Gelander and Monod is equivalent to the property that, whenever an  $L^p$ -representation of  $\Gamma$  admits a net of almost invariant unit vectors, it has a non-zero invariant vector. Central in the proof is to show that the closure of the group of  $\mathbb{T}$ -valued 1-coboundaries is a sufficient criteria for strong ergodicity of ergodic p.m.p. actions.

### III.1 Introduction

Kazhdan's property  $(T)$  is a rigidity property concerning how a group may act on a Hilbert space. It was first introduced by Kazhdan in 1967 in [Kaz67] and has since then become an important notion in analytic group theory. We refer to [BLV08] for a thorough introduction to the topic. In their seminal paper [BFGM07] from 2007, Bader, Furman, Gelander and Monod brought the notion of property  $(T)$  to the broader framework of Banach spaces. Their property  $(T_{\mathcal{E}})$ , where  $\mathcal{E}$  is a class of Banach spaces, is a rigidity property concerning how a group may act on spaces in the class  $\mathcal{E}$ . Since then, many authors have studied rigidity for actions on Banach spaces – see, e.g., [BO14], [Sal15], [LS21], [LO21], [MS23], [Opp23a] and [Opp23b], to name a few.

Let  $\Gamma$  be a discrete group. For a Banach space  $E$ , we denote by  $\text{Isom}(E)$  the group of linear surjective isometries on  $E$ . An isometric representation  $(\pi, E)$  of  $\Gamma$  on  $E$  is a group homomorphism  $\pi : \Gamma \rightarrow \text{Isom}(E)$ . When  $E$  is a Hilbert space, an isometric representation is commonly known as a unitary representation. Given an

isometric representation  $(\pi, E)$ , a vector  $\xi \in E$  is said to be *invariant* if  $\pi(t)\xi = \xi$ , for all  $t \in \Gamma$ . It is easy to verify that the set of invariant vectors forms a subspace of  $E$ , which we denote  $E^\pi$ . A net of vectors  $(\xi_i)_{i \in I}$  is said to be *almost invariant* if  $\|\pi(t)\xi_i - \xi_i\|_E \rightarrow 0$ , for every  $t \in \Gamma$ .

**Definition A** (Property  $(T_{\mathcal{E}})$ ). Let  $\mathcal{E}$  be a class of Banach spaces. A discrete group  $\Gamma$  has *property  $(T_{\mathcal{E}})$*  if, whenever an isometric representation  $(\pi, E)$  of  $\Gamma$  with  $E$  in the class  $\mathcal{E}$  admits a net of almost invariant unit vectors  $(\xi_i)_{i \in I}$ , there exists a net of invariant vectors  $(\eta_i)_{i \in I}$  such that  $\|\xi_i - \eta_i\|_E \rightarrow 0$ .

In [BFGM07], property  $(T_{\mathcal{E}})$  is defined in terms of the lack of nets of almost invariant unit vectors in the quotient  $E/E^\pi$ . By Lemma 18 in [Tan17], Definition A above is equivalent to the definition of Bader, Furman, Gelander and Monod. When  $\mathcal{E}$  is the class of complex Hilbert spaces, we recover Kazhdan's property  $(T)$ .

In the classical setting of unitary representations on Hilbert spaces, it is well-known that Kazhdan's property  $(T)$  allows several equivalent formulations. In particular, property  $(T)$  is often defined as the property that the existence of almost invariant unit vectors forces the existence of a non-zero invariant vector. When generalizing this property to the setting of actions on Banach spaces, we obtain an a priori weaker version of property  $(T_{\mathcal{E}})$ .

**Definition B** (Weak property  $(T_{\mathcal{E}})$ ). Let  $\mathcal{E}$  be a class of Banach spaces. A discrete group  $\Gamma$  has *weak property  $(T_{\mathcal{E}})$*  if any isometric representation  $(\pi, E)$  of  $\Gamma$  with  $E$  in the class  $\mathcal{E}$  admitting a net of almost invariant unit vectors has a non-zero invariant vector.

While it is easy to see that weak property  $(T_{\mathcal{E}})$  is implied by property  $(T_{\mathcal{E}})$ , the converse implication depends on the class  $\mathcal{E}$ . It is well-known to experts that the following two conditions are sufficient to ensure the equivalence of weak property  $(T_{\mathcal{E}})$  and property  $(T_{\mathcal{E}})$ :

- (i)  $\mathcal{E}$  is stable under quotients,
- (ii)  $\mathcal{E}$  is a class of superreflexive Banach spaces stable under taking complemented subspaces.

A proof of this can be found in Proposition 2.20 in [EP24]. Each of these conditions cover the case where  $\mathcal{E}$  is the class of Hilbert spaces. But interestingly, the class  $L^p$  of  $L^p$ -spaces on  $\sigma$ -finite measure spaces, for  $1 \leq p < \infty$ , does not satisfy either of the two conditions unless  $p = 2$ . In this paper, we address the question if weak property  $(T_{L^p})$  is the same as property  $(T_{L^p})$ . We show that the answer is affirmative when the group in question is countable discrete.

**Theorem C** (Theorem III.4.3). *A countable discrete group has property  $(T_{L^p})$  if and only if it has weak property  $(T_{L^p})$ .*

The proof of Theorem C relies on an analysis of ergodicity of measure preserving actions on probability spaces (in short: p.m.p. actions). The connection to property  $(T)$  is given by the characterization by Connes and Weiss in [CW80]: A discrete group has property  $(T)$  if and only if every p.m.p. ergodic action (on

a diffuse standard probability space) is strongly ergodic. We establish, via an application of the open mapping theorem for Polish groups, that the closure in a natural topology of the group of  $\mathbb{T}$ -valued 1-coboundaries for a given ergodic p.m.p. action is a sufficient condition to ensure strong ergodicity. This may be of independent interest.

**Theorem D** (Theorem III.3.2). *Let  $\Gamma$  be a countable discrete group,  $(\Omega, \nu)$  a separable probability space and  $\Gamma \curvearrowright (\Omega, \nu)$  an ergodic p.m.p. action. If  $B^1(\sigma; \mathbb{T})$  is closed then  $\sigma$  is strongly ergodic.*

A fair point can be made that the sufficient conditions (i) and (ii) above for equivalence of weak property  $(T_{\mathcal{E}})$  and property  $(T_{\mathcal{E}})$  are stronger than needed. For example, it is not necessary to require that every quotient stays in the class  $\mathcal{E}$ . Instead, given an isometric representation  $(\pi, L^p(\Omega, \nu))$ , we are interested only in the quotient  $L^p(\Omega, \nu)/L^p(\Omega, \nu)^{\pi}$ . This raises the question if a proof of Theorem C is viable without appealing to Theorem D. We give a partial negative answer to this question in Theorem E. When  $\pi$  comes from an ergodic p.m.p. action, the quotient  $L^p(\Omega, \nu)/L^p(\Omega, \nu)^{\pi}$  is equivariantly and isometrically isomorphic to the dual of the subspace  $L_0^{p'}(\Omega, \nu)$  of functions with mean zero, where  $p'$  is the Hölder conjugate of  $p$ . We show that the subspace  $L_0^p(\Omega, \nu)$  in many cases is not isometrically isomorphic to an  $L^p$ -space on a  $\sigma$ -finite measure space, and so, neither is its dual.

**Theorem E** (Theorem III.5.5). *Let  $(\Omega, \nu)$  be a diffuse standard probability space and let  $1 \leq p < \infty$ ,  $p \notin 2\mathbb{N}$ . Then  $L_0^p(\Omega, \nu)$  is not isometrically isomorphic to an  $L^p$ -space on a  $\sigma$ -finite measure space.*

This paper is organized as follows: In section III.2 we cover the preliminaries on Polish groups, actions on measure spaces, ergodicity and strong ergodicity, and on the topological groups of  $\mathbb{T}$ -valued 1-cocycles and 1-coboundaries. In section III.3 we prove Theorem D and in section III.4 we prove Theorem C. Finally, in section III.5, we prove Theorem E.

## III.2 Preliminaries

**Polish groups** A topological space is said to be *Polish* if it is separable and completely metrizable. A topological group is said to be *Polish* if it is Polish as a topological space. We list here a few permanence properties for Polish groups that shall become useful to us later. See, e.g., section 3 of [Kec95] for a reference.

**Proposition III.2.1.** *A closed subgroup of a Polish group is Polish.*

**Proposition III.2.2.** *A countable product of Polish groups is a Polish group.*

A main tool in this paper is the open mapping theorem in the setting of Polish groups, which we state below in Theorem III.2.3. It follows directly from Effros' theorem, which is Theorem 2.1 in [Eff65] (see also [Anc87] and Theorem 2.2.2 in [BK96]).

**Theorem III.2.3** (Open mapping theorem for Polish groups). *Let  $G$  and  $H$  be Polish groups and let  $\Phi : G \rightarrow H$  be a continuous and surjective group homomorphism. Then  $\Phi$  is open.*

**Actions on measure spaces** An introduction to group actions on measure spaces can be found, e.g., in section A.6 of [BLV08]. We recall briefly the main definitions. Let  $\Gamma$  be a discrete group. Given a  $\sigma$ -finite measure space  $(\Omega, \nu)$ , we denote by  $\text{Aut}(\Omega, [\nu])$  the group of all bi-measurable transformations of  $\Omega$  that leave  $\nu$  quasi-invariant. A *measure class preserving action* of  $\Gamma$  on  $(\Omega, \nu)$  is a group homomorphism  $\sigma : \Gamma \rightarrow \text{Aut}(\Omega, [\nu])$ . We write  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  for the action of  $\Gamma$  on  $(\Omega, \nu)$  given by  $\sigma$ . For  $t \in \Gamma$  and  $\omega \in \Omega$ , we shall often write  $t.\omega$  instead of  $\sigma_t(\omega)$ . Further, we denote by  $t.\nu$  the push forward measure of  $\nu$  by  $\sigma_t$ . For each  $t \in \Gamma$ , we denote by  $\frac{dt.\nu}{d\nu}$  the Radon-Nikodym derivative of  $t.\nu$  with respect to  $\nu$ . The assumption that the action is measure class preserving ensures that this Radon-Nikodym derivative exists. Recall that it is a strictly positive function. Let  $L^0(\Omega, \nu)$  denote the space of (equivalence classes of) measurable complex-valued functions on  $(\Omega, \nu)$ . For  $\xi \in L^0(\Omega, \nu)$  and  $t \in \Gamma$ , we denote by  $t.\xi$  be the measurable function given by  $t.\xi(\omega) = \xi(t^{-1}.\omega)$ , for  $\omega \in \Omega$ . In this way, the action  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  induces in a canonical way an action of  $\Gamma$  on  $L^0(\Omega, \nu)$ .

When an action  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  leaves the measure  $\nu$  invariant rather than just quasi-invariant, we say that it is *measure preserving*. A measure preserving action on a probability space is called a *probability measure preserving action* (in short: a p.m.p. action). If the action is measure preserving,  $\frac{dt.\nu}{d\nu}$  is everywhere equal to 1, for all  $t \in \Gamma$ .

**Ergodicity and strong ergodicity** Let  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  be a measure class preserving action of a discrete group on a  $\sigma$ -finite measure space. A measurable subset  $A$  of  $\Omega$  is said to be  $\Gamma$ -invariant if  $\nu(t.A \Delta A) = 0$ , for all  $t \in \Gamma$ . Observe that null and co-null subsets are trivially  $\Gamma$ -invariant. We say that the action  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  is *ergodic* if there are no non-trivial  $\Gamma$ -invariant measurable subsets of  $\Omega$ .

When  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  is a p.m.p. action, a stronger version of ergodicity is defined as follows: A sequence  $(A_n)_{n \geq 1}$  of measurable subsets of  $\Omega$  is said to be *asymptotically  $\Gamma$ -invariant* if  $\nu(t.A_n \Delta A_n)$  converges to zero, for all  $t \in \Gamma$ . Given an asymptotically  $\Gamma$ -invariant sequence  $(A_n)_{n \geq 1}$ , we say that it is *non-trivial* if  $\liminf_n \nu(A_n) \nu(A_n^c) > 0$ . The action  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  is said to be *strongly ergodic* if there are no non-trivial asymptotically  $\Gamma$ -invariant sequences of measurable subsets of  $\Omega$ . It is clear that any strongly ergodic action is automatically ergodic. A deep result by Connes and Weiss in [CW80] (see also Theorem 6.3.4 in [BLV08]) shows that the converse implication characterizes groups with property (T). We state it here for discrete groups.

**Theorem III.2.4** (Connes–Weiss). *A discrete group  $\Gamma$  has property (T) if and only if every ergodic p.m.p. action is strongly ergodic.*

*Remark III.2.5.* In Theorem III.2.4, it is enough to consider actions on diffuse standard probability spaces, i.e., probability spaces that are isomorphic mod 0 to

the interval with the Lebesgue measure. This follows from the proof of Corollary A.7.15 in [BLV08] and Theorem 17.41 in [Kec95].

We refer to section 6.3 of [BLV08] for more background on ergodicity of group actions and a comprehensive review of the connection between ergodicity and property (T).

**The  $\mathbb{T}$ -valued measurable functions as a topological group** Let  $(\Omega, \nu)$  be a probability space. We denote by  $L^0(\Omega, \nu; \mathbb{T})$  the set of measurable functions on  $(\Omega, \nu)$  with values in  $\mathbb{T}$ . We have a natural group structure on  $L^0(\Omega, \nu; \mathbb{T})$  with multiplication defined pointwise. The multiplicative unit is the function  $1_\Omega$  which is everywhere equal to 1. The inverse of a function in  $L^0(\Omega, \nu; \mathbb{T})$  is its complex conjugate. We equip  $L^0(\Omega, \nu; \mathbb{T})$  with the topology of convergence in measure, i.e., the topology generated by the sets of the form

$$V_\varepsilon(\varphi_0) = \left\{ \varphi \in L^0(\Omega, \nu; \mathbb{T}) \mid \nu(\{|\varphi - \varphi_0| \geq \varepsilon\}) < \varepsilon \right\}, \quad (\text{III.1})$$

where  $\varphi_0 \in L^0(\Omega, \nu; \mathbb{T})$  and  $\varepsilon > 0$ . A standard computation verifies that this makes  $L^0(\Omega, \nu; \mathbb{T})$  a topological group. Lemma III.2.6 below is well-known to experts. It shows, in particular, that the topology on  $L^0(\Omega, \nu; \mathbb{T})$  is metrizable.

**Lemma III.2.6.** *Let  $(\Omega, \nu)$  be a probability space and fix  $1 \leq p < \infty$ . The topology on  $L^0(\Omega, \nu; \mathbb{T})$  generated by the sets in equation (III.1) is equivalent to the topology generated by the sets*

$$V_\varepsilon^p(\varphi_0) = \left\{ \varphi \in L^0(\Omega, \nu; \mathbb{T}) \mid \|\varphi - \varphi_0\|_p < \varepsilon \right\},$$

where  $\varphi_0 \in L^0(\Omega, \nu; \mathbb{T})$  and  $\varepsilon > 0$ .

*Proof.* Denote by  $\mathcal{T}_0$  the topology on  $L^0(\Omega, \nu; \mathbb{T})$  generated by the sets of the form  $V_\varepsilon(\varphi_0)$  and by  $\mathcal{T}_p$  the topology generated by the sets of the form  $V_\varepsilon^p(\varphi_0)$ . Fix  $\varphi_0 \in L^0(\Omega, \nu; \mathbb{T})$  and  $\varepsilon > 0$ . If  $\varphi \in L^0(\Omega, \nu; \mathbb{T})$  is such that  $\|\varphi - \varphi_0\|_p^p < \varepsilon^{p+1}$  then

$$\nu(\{|\varphi - \varphi_0| \geq \varepsilon\}) \leq \frac{1}{\varepsilon^p} \int_{\{|\varphi - \varphi_0| \geq \varepsilon\}} |\varphi - \varphi_0|^p \, d\nu \leq \frac{1}{\varepsilon^p} \int_\Omega |\varphi - \varphi_0|^p \, d\nu < \varepsilon.$$

Hence,  $V_{\varepsilon^{1+1/p}}^p(\varphi_0) \subset V_\varepsilon(\varphi_0)$ . This shows that  $\mathcal{T}_p$  is finer than  $\mathcal{T}_0$ . Conversely, if  $\varphi \in L^0(\Omega, \nu; \mathbb{T})$  is such that  $\nu(\{|\varphi - \varphi_0| \geq \varepsilon\}) < \varepsilon$  then

$$\begin{aligned} \int_\Omega |\varphi - \varphi_0|^p \, d\nu &= \int_{\{|\varphi - \varphi_0| \geq \varepsilon\}} |\varphi - \varphi_0|^p \, d\nu + \int_{\{|\varphi - \varphi_0| < \varepsilon\}} |\varphi - \varphi_0|^p \, d\nu \\ &\leq 2^p \nu(\{|\varphi - \varphi_0| \geq \varepsilon\}) + \varepsilon^p \nu(\{|\varphi - \varphi_0| < \varepsilon\}) \\ &< \varepsilon 2^p + \varepsilon^p. \end{aligned}$$

Hence,  $V_\varepsilon(\varphi_0) \subset V_{(\varepsilon 2^p + \varepsilon^p)^{1/p}}^p(\varphi_0)$ . Since  $(\varepsilon 2^p + \varepsilon^p)^{1/p} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , this shows that  $\mathcal{T}_0$  is finer than  $\mathcal{T}_p$ . Hence, the two topologies are equivalent.  $\square$

Recall that a measure space  $(\Omega, \nu)$  is said to be separable if the space of measurable subsets of  $\Omega$  is separable as a topological space with respect to the distance given by  $\nu(A \Delta B)$ , for  $A, B \subset \Omega$  measurable.

**Proposition III.2.7.** *Let  $(\Omega, \nu)$  be a separable probability space. Then  $L^0(\Omega, \nu; \mathbb{T})$  is a Polish group with multiplication defined pointwise and with the topology generated by the sets defined in equation (III.1).*

*Proof.* Because  $\nu$  is finite, any measurable function with values in  $\mathbb{T}$  is integrable. Hence, we can view  $L^0(\Omega, \nu; \mathbb{T})$  as a subset of  $L^1(\Omega, \nu)$ . By Lemma III.2.6, the subspace topology on  $L^0(\Omega, \nu; \mathbb{T})$  coming from this embedding agrees with the topology generated by the sets defined in equation (III.1). It is a standard computation to verify that  $L^0(\Omega, \nu; \mathbb{T})$  is closed in  $L^1(\Omega, \nu)$ . Hence,  $L^0(\Omega, \nu; \mathbb{T})$  is completely metrizable. Finally, as  $(\Omega, \nu)$  is separable,  $L^1(\Omega, \nu)$  is separable and then so is  $L^0(\Omega, \nu; \mathbb{T})$ .  $\square$

**Groups of 1-cocycles and 1-coboundaries** Let  $\Gamma$  be a discrete group and  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  a measure class preserving action. A  $(\mathbb{T}$ -valued) 1-cocycle for the action  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  is a map  $c : \Gamma \times \Omega \rightarrow \mathbb{T}$  such that  $c_t$  is a measurable map, for every  $t \in \Gamma$ , and such that, for every pair  $s, t \in \Gamma$  and almost every  $\omega \in \Omega$ ,

$$c_{st}(\omega) = c_s(\omega)c_t(s^{-1}.\omega).$$

The set of all 1-cocycles is denoted by  $Z^1(\sigma; \mathbb{T})$ . Given  $\varphi \in L^0(\Omega, \nu; \mathbb{T})$ , define a map  $b_\varphi : \Gamma \times \Omega \rightarrow \mathbb{T}$  by

$$b_\varphi(t, \omega) = \frac{\varphi(\omega)}{\varphi(t^{-1}.\omega)}, \quad (\text{III.2})$$

for  $t \in \Gamma$  and  $\omega \in \Omega$ . It is straight forward to verify that  $b_\varphi$  is a 1-cocycle. A 1-cocycle of this form is called a 1-coboundary. The set of all 1-coboundaries is denoted by  $B^1(\sigma; \mathbb{T})$ .

We equip  $Z^1(\sigma; \mathbb{T})$  with a group structure as follows: Given two 1-cocycles  $c, d \in Z^1(\sigma; \mathbb{T})$ , their product is defined via the multiplication on  $\mathbb{T}$  by setting  $(c \cdot d)_t(\omega) = c_t(\omega)d_t(\omega)$ , for  $t \in \Gamma$  and  $\omega \in \Omega$ . Because  $\mathbb{T}$  is an abelian group,  $c \cdot d$  is again a 1-cocycle, and so,  $\cdot$  gives a well-defined multiplication on  $Z^1(\sigma; \mathbb{T})$ . The 1-coboundary  $b_{1_\Omega}$  is the multiplicative identity. The inverse of a 1-cocycle is given by its complex conjugate. Further, we have a canonical embedding of groups  $Z^1(\sigma; \mathbb{T}) \hookrightarrow L^0(\Omega, \nu; \mathbb{T})^\Gamma$  given by mapping  $c \in Z^1(\sigma; \mathbb{T})$  to the  $\Gamma$ -indexed sequence  $(c_t)_{t \in \Gamma}$ . This embedding gives  $Z^1(\sigma; \mathbb{T})$  the structure of a topological group inheriting the product topology from  $L^0(\Omega, \nu; \mathbb{T})^\Gamma$ . It can be verified with a standard computation that, as such, it is closed. We include a proof in Proposition III.2.8 below that  $Z^1(\sigma; \mathbb{T})$  is Polish under the additional assumptions that  $\Gamma$  is countable and  $(\Omega, \nu)$  is separable. This fact can be found without proof in section 24 of [Kec10].

**Proposition III.2.8.** *Let  $\Gamma$  be a countable discrete group and let  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  be a measure class preserving action on a separable probability space. Then  $Z^1(\sigma; \mathbb{T})$  is a Polish group.*

*Proof.* When  $\Gamma$  is countable discrete and  $(\Omega, \nu)$  is a separable probability space,  $L^0(\Omega, \nu; \mathbb{T})^\Gamma$  is a Polish group by Proposition III.2.7 and Proposition III.2.2. Because  $Z^1(\sigma; \mathbb{T})$  is closed in  $L^0(\Omega, \nu; \mathbb{T})^\Gamma$ , it follows from Proposition III.2.1 that  $Z^1(\sigma; \mathbb{T})$  is a Polish group.  $\square$

We equip  $B^1(\sigma; \mathbb{T})$  with the subspace topology. When  $B^1(\sigma; \mathbb{T})$  is closed, it is Polish. But this need not be the case. In particular, we shall see in the proof of Theorem D that  $B^1(\sigma; \mathbb{T})$  is not Polish when  $\sigma$  is an ergodic but not strongly ergodic p.m.p. action. By the characterization of property (T) by Connes and Weiss (Theorem III.2.4), every group without property (T) admits such an action.

For more background on 1-cocycles and 1-coboundaries, we refer to chapter 3 of [Kec10].

### III.3 Proof of Theorem D

In this section, we prove Theorem D from the introduction (Theorem III.3.2 below), which gives a connection between the closure of the space of  $\mathbb{T}$ -valued 1-coboundaries and strong ergodicity of the action. The main tool in the proof is an application of the open mapping theorem for Polish groups (Theorem III.2.3) to the map  $\beta : L^0(\Omega, \nu; \mathbb{T}) \rightarrow B^1(\sigma; \mathbb{T})$  given by  $\beta(\varphi) = b_\varphi$ , where  $b_\varphi$  is as defined equation (III.2).

**Lemma III.3.1.** *Let  $\Gamma$  be a discrete group and let  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  be an ergodic p.m.p. action. For each  $\varphi \in L^0(\Omega, \nu; \mathbb{T})$ , let  $b_\varphi \in B^1(\sigma; \mathbb{T})$  be as in (III.2). The map  $\beta : L^0(\Omega, \nu; \mathbb{T}) \rightarrow B^1(\sigma; \mathbb{T})$  given by  $\beta(\varphi) = b_\varphi$  is a continuous and surjective group homomorphism whose kernel is the subgroup of constant functions.*

*Proof.* For  $\varphi, \psi \in L^0(\Omega, \nu; \mathbb{T})$ ,  $t \in \Gamma$  and  $\omega \in \Omega$ , we have

$$b_{\varphi\psi}(t, \omega) = \frac{\varphi(\omega)\psi(\omega)}{\varphi(t^{-1}\cdot\omega)\psi(t^{-1}\cdot\omega)} = b_\varphi(t, \omega)b_\psi(t, \omega).$$

Hence,  $\beta$  is a group homomorphism. It is surjective by definition of  $B^1(\sigma; \mathbb{T})$ . To see that  $\beta$  is continuous, it suffices to show that it is sequentially continuous since the topology on  $L^0(\Omega, \nu; \mathbb{T})$  is metrizable by Lemma III.2.6. Let  $\varphi_n$  be a convergent sequence in  $L^0(\Omega, \nu; \mathbb{T})$  with limit  $\varphi$ . Then  $\|\varphi - \varphi_n\|_1 \rightarrow 0$ , by Lemma III.2.6. For every  $t \in \Gamma$ , we have

$$\|b_\varphi(t, \square) - b_{\varphi_n}(t, \square)\|_1 \leq \|\varphi - \varphi_n\|_1 + \|t\cdot\varphi - t\cdot\varphi_n\|_1 = 2\|\varphi - \varphi_n\|_1 \rightarrow 0.$$

In the last equality, we use that the measure is invariant for the action of  $\Gamma$ . Because  $\Gamma$  is discrete, it follows that  $b_{\varphi_n} \rightarrow b_\varphi$  in  $B^1(\sigma; \mathbb{T})$ . Hence,  $\beta$  is continuous. Finally,  $\ker \beta$  consists of functions which are constant on the orbits of  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$ . Since the action is ergodic, it follows that  $\ker \beta$  is the subgroup of constant functions.  $\square$

**Theorem III.3.2.** *Let  $\Gamma$  be a countable discrete group,  $(\Omega, \nu)$  a separable probability space and  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  an ergodic p.m.p. action. If  $B^1(\sigma; \mathbb{T})$  is closed then  $\sigma$  is strongly ergodic.*

*Proof.* Because  $\Gamma$  is countable discrete and  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  is a p.m.p. action on a separable probability space,  $Z^1(\sigma; \mathbb{T})$  is Polish by Proposition III.2.8. Assume  $B^1(\sigma; \mathbb{T})$  is closed. Then it is a Polish group, by Proposition III.2.1. Because the map  $\beta : L^0(\Omega, \nu; \mathbb{T}) \rightarrow B^1(\sigma; \mathbb{T})$  from Lemma III.3.1 is then a continuous,

surjective homomorphism of Polish groups, we deduce from Theorem III.2.3 that  $\beta$  is open. Hence, for each  $\varepsilon > 0$ , the set

$$\beta(V_\varepsilon(1_\Omega)) = \left\{ b_\varphi \mid \varphi \in L^0(\Omega, \nu; \mathbb{T}) \text{ such that } \nu(\{|1_\Omega - \varphi| \geq \varepsilon\}) < \varepsilon \right\}$$

is an open set in  $B^1(\sigma; \mathbb{T})$  containing  $b_{1_\Omega}$ . By construction of the topology on  $B^1(\sigma; \mathbb{T})$ , there is a finite subset  $F_\varepsilon \subset \Gamma$  (depending on  $\varepsilon$ ) and, for each  $t \in F_\varepsilon$ , an open subset of  $L^0(\Omega, \nu; \mathbb{T})$  such that the product of these inside  $L^0(\Omega, \nu; \mathbb{T})^\Gamma$  intersected with  $B^1(\sigma; \mathbb{T})$  is contained in  $\beta(V_\varepsilon(1_\Omega))$ . Employing Lemma III.2.6, we can choose the open subsets of  $L^0(\Omega, \nu; \mathbb{T})$  to be of the form  $V_\delta^1(1_\Omega)$ , for some  $\delta > 0$ . That is, for each  $\varepsilon > 0$ , we can take a finite subset  $F_\varepsilon \subset \Gamma$  and a  $\delta_\varepsilon > 0$  such that the set

$$\left\{ b \in B^1(\sigma; \mathbb{T}) \mid \|b(t, \square) - 1_\Omega\|_1 < \delta_\varepsilon \text{ for every } t \in F_\varepsilon \right\}$$

is contained in  $\beta(V_\varepsilon(1_\Omega))$ . Hence, if  $b \in B^1(\sigma; \mathbb{T})$  is any 1-coboundary satisfying that  $\|b(t, \square) - 1_\Omega\|_1 < \delta_\varepsilon$ , for every  $t \in F_\varepsilon$ , then there is a  $\psi \in L^0(\Omega, \nu; \mathbb{T})$  such that  $b = b_\psi$  and such that  $\nu(\{|1_\Omega - \psi| \geq \varepsilon\}) < \varepsilon$ .

Let  $(A_n)_{n \geq 1}$  be an asymptotically invariant sequence of measurable subsets of  $\Omega$  and set, for each  $n \in \mathbb{N}$ ,  $\varphi_n = 2 \cdot 1_{A_n} - 1_\Omega$  and  $b_n = b_{\varphi_n} \in B^1(\sigma; \mathbb{T})$ . Then, for each  $t \in \Gamma$ ,

$$\|b_n(t, \square) - 1_\Omega\|_1 = \|\varphi_n - t \cdot \varphi_n\|_1 = 2 \|1_{A_n} - t \cdot 1_{A_n}\|_1 = 2\nu(A_n \Delta t \cdot A_n) \rightarrow 0.$$

Let  $(N_k)_{k \geq 1}$  be a strictly increasing sequence in  $\mathbb{N}$  such that  $\|b_{N_k}(t, \square) - 1_\Omega\|_1 < \delta_{1/k}$ , for each  $t \in F_{1/k}$  and for every  $k \in \mathbb{N}$ . For each  $k \in \mathbb{N}$ , we can then find  $\psi_k \in L^0(\Omega, \nu; \mathbb{T})$  such that  $b_{N_k} = b_{\psi_k}$  and such that  $\nu(\{|1_\Omega - \psi_k| \geq \frac{1}{k}\}) < \frac{1}{k}$ . Because  $\ker \beta \cong \mathbb{T}$ , by Lemma III.3.1, we must have  $\psi_k = e^{i\theta_k} \varphi_{N_k}$ , for some  $\theta_k \in [0, 2\pi)$ . Observe that

$$(1_\Omega - e^{i\theta_k} \varphi_{N_k})(\omega) = \begin{cases} 1 - e^{i\theta_k} & \text{if } \omega \in A_{N_k} \\ 1 + e^{i\theta_k} & \text{if } \omega \notin A_{N_k} \end{cases}.$$

The two complex numbers  $1 - e^{i\theta_k}$  and  $1 + e^{i\theta_k}$  lie on opposite sides of a unit circle around 1, and so, at least one of them will have modulus greater than 1. As  $\psi_k$  satisfies  $\nu(\{|1_\Omega - \psi_k| \geq \frac{1}{k}\}) < \frac{1}{k}$ , the other one must have modulus strictly less than  $1/k$ . This gives us two cases: If  $|1 - e^{i\theta_k}| < 1/k$  then

$$\nu\left(\left\{|1_\Omega - \psi_k| \geq \frac{1}{k}\right\}\right) = \nu(A_{N_k}^c).$$

Otherwise, if  $|1 + e^{i\theta_k}| < 1/k$  then

$$\nu\left(\left\{|1_\Omega - \psi_k| \geq \frac{1}{k}\right\}\right) = \nu(A_{N_k}).$$

Hence, for each  $k \in \mathbb{N}$ , it is either the case that  $\nu(A_{N_k}) < \frac{1}{k}$  or that  $\nu(A_{N_k}^c) < \frac{1}{k}$ . Then, for each  $k \in \mathbb{N}$ ,

$$\nu(A_{N_k}) \nu(A_{N_k}^c) \leq \min\left\{\nu(A_{N_k}), \nu(A_{N_k}^c)\right\} < \frac{1}{k}.$$

It follows that

$$\liminf_{n \in \mathbb{N}} \nu(A_n) \nu(A_n^c) = 0.$$

Hence,  $(A_n)_{n \geq 1}$  is not non-trivial. Since  $(A_n)_{n \geq 1}$  was an arbitrary asymptotically  $\Gamma$ -invariant sequence, it follows that  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  is strongly ergodic.  $\square$

The characterization of property (T) by Connes and Weiss (Theorem III.2.4) together with Theorem III.3.2 immediately implies the following corollary:

**Corollary III.3.3.** *Let  $\Gamma$  be a discrete group. If  $B^1(\sigma; \mathbb{T})$  is closed in  $Z^1(\sigma; \mathbb{T})$ , for every ergodic p.m.p. action  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$ , then  $\Gamma$  has property (T).*

*Remark III.3.4.* By Remark III.2.5, it suffices in Corollary III.3.3 to consider actions on diffuse standard probability spaces.

## III.4 Proof of Theorem C

Let  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  be an ergodic measure class preserving action of a discrete group on a separable  $\sigma$ -finite measure space. Let  $1 \leq p < \infty$  and  $c \in Z^1(\sigma; \mathbb{T})$  be given. For each  $t \in \Gamma$  and  $\xi \in L^p(\Omega, \nu)$ , set

$$\pi_{p,\sigma,c}(t)\xi(\omega) = c_t(\omega) \left( \frac{dt.\nu}{d\nu} \right)^{1/p}(\omega) \xi(t^{-1}.\omega).$$

Then  $\pi_{p,\sigma,c}$  is an isometric representation of  $\Gamma$  on  $L^p(\Omega, \nu)$ . We refer to, e.g., [Gar21] for a modern review of group representations on  $L^p$ -spaces. In this section, we characterize when  $\pi_{p,\sigma,c}$  has invariant vectors (see Proposition III.4.1). Moreover, we give a sufficient condition for the existence of almost invariant unit vectors in the setting where the action is measure preserving (see Proposition III.4.2). Together with the connection between property (T) and the closure of the space of 1-coboundaries shown in Corollary III.3.3, these insights allow us to show the equivalence of weak property  $(T_{L^p})$  and property  $(T_{L^p})$ . This is Theorem C in the introduction and Theorem III.4.3 below.

**Proposition III.4.1.** *Let  $\Gamma$  be a discrete group and  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  an ergodic measure class preserving action on a  $\sigma$ -finite measure space. For  $c \in Z^1(\sigma; \mathbb{T})$  and  $1 \leq p < \infty$ , denote by  $\pi_{p,\sigma,c}$  the associated representation of  $\Gamma$  on  $L^p(\Omega, \nu)$ . Then  $\pi_{p,\sigma,c}$  admits a non-zero invariant vector if and only if  $c$  is a 1-coboundary and  $\nu$  is equivalent to a finite  $\Gamma$ -invariant measure.*

*Proof.* Suppose  $\xi \in L^p(\Omega, \nu)$  is a non-zero invariant vector for  $\pi_{p,\sigma,c}$ . Then, for every  $t \in \Gamma$ ,

$$\xi = c_t \left( \frac{dt.\nu}{d\nu} \right)^{1/p} t.\xi \quad \nu\text{-a.e.}, \quad (\text{III.3})$$

where  $t.\xi$  is the function given by  $t.\xi(\omega) = \xi(t^{-1}.\omega)$ , for  $\omega \in \Omega$ . For each  $t \in \Gamma$ , let  $\Omega_0^t \subset \Omega$  be the subset where we have equality in (III.3) and set  $\Omega_0 = \bigcap_{t \in \Gamma} \Omega_0^t$ . Then  $\Omega_0$  is measurable and co-null. Because  $c_t$  takes values in  $\mathbb{T}$ , we see that

$$|\xi| = \left( \frac{dt.\nu}{d\nu} \right)^{1/p} |t.\xi| \quad \nu\text{-a.e.}, \quad (\text{III.4})$$

for every  $t \in \Gamma$  and with equality on  $\Omega_0$ . Then, since the Radon-Nikodym derivative is strictly positive, it follows that the set  $\{\xi = 0\} \cap \Omega_0$  is  $\Gamma$ -invariant. Because  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  is ergodic and  $\xi$  is non-zero, this implies that  $\nu(\{\xi = 0\}) = 0$ . We then get a  $\nu$ -almost everywhere uniquely defined measurable function  $\varphi \in L^0(\Omega, \nu; \mathbb{T})$  such that  $\xi = \varphi |\xi|$ . Insert this into equation (III.3) and apply equation (III.4) to obtain, for every  $t \in \Gamma$ ,

$$\varphi |\xi| = c_t \left( \frac{dt.\nu}{d\nu} \right)^{1/p} t.\varphi |t.\xi| = c_t t.\varphi |\xi| \quad \nu\text{-a.e.},$$

with equality on  $\Omega_0$ . Since,  $\nu(\{\xi = 0\}) = 0$ , we deduce that, for every  $t \in \Gamma$ ,

$$c_t = \frac{\varphi}{t.\varphi} \quad \nu\text{-a.e.}.$$

Hence,  $c$  is a 1-coboundary. Further, for every  $t \in \Gamma$  and every measurable subset  $B \subset \Omega$ , we have

$$\begin{aligned} \int_{\Omega} 1_{t.B} |\xi|^p d\nu &= \int_{\Omega} t.1_B \frac{dt.\nu}{d\nu} |t.\xi|^p d\nu \\ &= \int_{\Omega} 1_B |\xi|^p d\nu. \end{aligned}$$

Here, we have used equation (III.4) in the first equality and the change of variable formula in the second. This shows that the finite measure  $|\xi|^p d\nu$  is  $\Gamma$ -invariant. Because  $\nu(\{\xi = 0\}) = 0$ , we see that  $|\xi|^p d\nu$  is equivalent to  $\nu$ .

Conversely, suppose that  $\nu$  is equivalent to a finite  $\Gamma$ -invariant measure  $\mu$ , and that  $c$  is a 1-coboundary. Write  $c = b_\varphi$ , for a  $\varphi \in L^0(\Omega, \nu; \mathbb{T})$ . Since  $\mu$  is finite, the Radon-Nikodym derivative  $\frac{d\mu}{d\nu}$  is  $\nu$ -integrable. Set  $\xi = \varphi \cdot \left(\frac{d\mu}{d\nu}\right)^{1/p}$ . Then  $\xi$  is non-zero and lies in  $L^p(\Omega, \nu)$ . We have, for every measurable subset  $B \subset \Omega$ , the equality

$$\int_{\Omega} 1_{t.B} |\xi|^p d\nu = \mu(t.B) = \mu(B) = \int_{\Omega} 1_B |\xi|^p d\nu.$$

Further, for each  $t \in \Gamma$ ,

$$\int_{\Omega} 1_B |\xi|^p d\nu = \int_{\Omega} 1_{t.B} \frac{dt.\nu}{d\nu} |t.\xi|^p d\nu,$$

by the change of variable formula. Putting this together, we see that

$$\int_{\Omega} 1_{t.B} \left( |\xi|^p - \frac{dt.\nu}{d\nu} |t.\xi|^p \right) d\nu = 0,$$

for every measurable subset  $B \subset \Omega$  and every  $t \in \Gamma$ . In particular, this equality holds for the following measurable subsets of  $\Omega$ :

$$t^{-1} \cdot \left\{ |\xi|^p > \frac{dt \cdot \nu}{d\nu} |t \cdot \xi|^p \right\} \quad \text{and} \quad t^{-1} \cdot \left\{ |\xi|^p < \frac{dt \cdot \nu}{d\nu} |t \cdot \xi|^p \right\}.$$

Hence,

$$|\xi|^p = \frac{dt \cdot \nu}{d\nu} |t \cdot \xi|^p \quad \nu\text{-a.e.}$$

We deduce that

$$\pi_{p,\sigma,b_\varphi}(t)\xi = \frac{\varphi}{t \cdot \varphi} \left( \frac{dt \cdot \nu}{d\nu} \right)^{1/p} t \cdot \xi = \varphi \left( \frac{dt \cdot \nu}{d\nu} \right)^{1/p} |t \cdot \xi| = \varphi |\xi| = \xi \quad \nu\text{-a.e.}$$

That is,  $\xi$  is a non-zero invariant vector for  $\pi_{p,\sigma,b_\varphi}$ .  $\square$

**Proposition III.4.2.** *Let  $\Gamma$  be a discrete group, let  $(\Omega, \nu)$  be a probability space, let  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  be a p.m.p. action and let  $1 \leq p < \infty$ . If  $c \in Z^1(\sigma; \mathbb{T})$  is the limit of a net of 1-coboundaries then  $\pi_{p,\sigma,c}$  admits a net of almost invariant unit vectors.*

*Proof.* Let  $(\varphi_i)_i$  be a net in  $L^0(\Omega, \nu; \mathbb{T})$ . For each index  $i$ , write  $b_i = b_{\varphi_i}$  for the associated 1-coboundary. Assume that  $(\varphi_i)_i$  is such that  $c$  is the limit of  $(b_i)_i$ . For each index  $i$ ,  $\varphi_i$  lies in  $L^p(\Omega, \nu)$  with unit norm since  $(\Omega, \nu)$  is a probability space and  $\varphi_i$  has everywhere modulus equal to 1. We have, for all  $t \in \Gamma$ ,

$$\|\pi_{p,\sigma,c}(t)\varphi_i - \varphi_i\|_p = \|c_t t \cdot \varphi_i - \varphi_i\|_p = \|c_t - b_i(t, \square)\|_p.$$

Because  $p$  is finite and because  $c$  is the limit of  $(b_i)_i$ , this converges to zero by Lemma III.2.6. Hence,  $(\varphi_i)_i$  is a net of almost invariant unit vectors for  $\pi_{p,\sigma,c}$ .  $\square$

**Theorem III.4.3.** *A discrete group has property  $(T_{L^p})$  if and only if it has weak property  $(T_{L^p})$ .*

*Proof.* Let  $\Gamma$  be a discrete group without property  $(T_{L^p})$ . Then  $\Gamma$  does not have property  $(T)$ , by Theorem A in [BFGM07]. Hence, by Corollary III.3.3 and Remark III.3.4, there is an ergodic p.m.p. action  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  on a separable probability space such that  $B^1(\sigma; \mathbb{T})$  is not closed. We can then find a 1-cocycle  $c \in Z^1(\sigma, \mathbb{T})$  which is not a 1-coboundary but which is the limit of a net of 1-coboundaries. It follows from Proposition III.4.1 and Proposition III.4.2 that  $\pi_{p,\sigma,c}$  has almost invariant unit vectors but no non-zero invariant vector.  $\square$

## III.5 On the possibility of an easier proof

The class  $L^p$ , for  $1 \leq p < \infty$  and  $p \neq 2$ , is neither stable under quotients nor under complemented subspaces. Therefore, it does not meet either of the conditions of Proposition 2.20 in [EP24] for the equivalence of weak property  $(T_{\mathcal{E}})$  and property  $(T_{\mathcal{E}})$ . However, as mentioned in the introduction, these conditions are stronger than needed. In this section, we shall address the question if a proof of Theorem C is possible via the ideas used in [EP24]. Precisely, given

an isometric  $L^p$ -representation  $(\pi, L^p(\Omega, \nu))$ , if one could guarantee the existence of another isometric  $L^p$ -representation  $(\rho, L^p(\Omega', \mu))$  and a bounded isomorphism  $\Phi : L^p(\Omega, \nu)/L^p(\Omega, \nu)^\pi \rightarrow L^p(\Omega', \mu)$  such that  $\rho = \Phi \circ \pi \circ \Phi^{-1}$ , the equivalence of weak property  $(T_{L^p})$  and property  $(T_{L^p})$  would follow directly. In Theorem III.5.5, which is Theorem E in the introduction, we give an example of an isometric representation on an  $L^p$ -space where the quotient with the subspace of invariant vectors is not isometrically isomorphic to any  $L^p$ -space on a  $\sigma$ -finite measure space.

Let  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  be an ergodic p.m.p. action and denote by  $\pi_{p,\sigma}$  the associated representation on  $L^p(\Omega, \nu)$  (with the trivial 1-cocycle). Observe that  $L^p(\Omega, \nu)^{\pi_{p,\sigma}} \cong \mathbb{C}1_\Omega$ , and so, the quotient  $L^p(\Omega, \nu)/L^p(\Omega, \nu)^{\pi_{p,\sigma}}$  is isomorphic as a vector space to the  $\Gamma$ -invariant subspace of functions with mean zero:

$$L_0^p(\Omega, \nu) = \left\{ f \in L^p(\Omega, \nu) \mid \int_\Omega f \, d\nu = 0 \right\}.$$

Denote by  $\pi_{p,\sigma}^0$  the restriction of  $\pi_{p,\sigma}$  to  $L_0^p(\Omega, \nu)$  and by  $\bar{\pi}_{p,\sigma}$  the representation on the quotient  $L^p(\Omega, \nu)/L^p(\Omega, \nu)^{\pi_{p,\sigma}}$  coming from  $\pi_{p,\sigma}$ . A straight forward computation confirms that the isomorphism between  $L^p(\Omega, \nu)/L^p(\Omega, \nu)^{\pi_{p,\sigma}}$  and  $L_0^p(\Omega, \nu)$  is equivariant, i.e., that it intertwines  $\pi_{p,\sigma}^0$  and  $\bar{\pi}_{p,\sigma}$ . However, it need not be isometric. Proposition III.5.1 below is well-known to experts.

**Proposition III.5.1.** *Let  $\Gamma \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  be an ergodic p.m.p. action, and let  $1 < p, p' < \infty$  be Hölder conjugates. The quotient  $L^p(\Omega, \nu)/L^p(\Omega, \nu)^{\pi_{p,\sigma}}$  is equivariantly and isometrically isomorphic to the dual of  $L_0^{p'}(\Omega, \nu)$ .*

*Proof.* For  $f \in L_0^p(\Omega, \nu)$  and  $g \in L_0^{p'}(\Omega, \nu)$ , set  $\varphi_f(g) = \int_\Omega f g \, d\nu$ . Then  $\varphi_f$  is a linear functional on  $L_0^{p'}(\Omega, \nu)$  and it is straightforward to check that  $f \mapsto \varphi_f$  defines a vector space isomorphism  $\varphi : L_0^p(\Omega, \nu) \rightarrow L_0^{p'}(\Omega, \nu)'$ . Denote by  $(\pi_{p',\sigma}^0)'$  the dual representation of the restriction of  $\pi_{p',\sigma}$  to  $L_0^{p'}(\Omega, \nu)$ . For each  $f \in L_0^p(\Omega, \nu)$  and  $g \in L_0^{p'}(\Omega, \nu)$  and for each  $t \in \Gamma$ ,

$$(\pi_{p',\sigma}^0)'(t)\varphi_f(g) = \varphi_f(\pi_{p,\sigma}^0(t^{-1})g) = \int_\Omega g(t.\omega)f(\omega) \, d\nu(\omega) = \varphi_{\pi_{p,\sigma}(t)f}(g),$$

where we have used invariance of  $\nu$  in the last equality. This shows that  $\varphi$  is equivariant. Precomposing  $\varphi$  with the canonical equivariant isomorphism from  $L^p(\Omega, \nu)/L^p(\Omega, \nu)^{\pi_{p,\sigma}}$  to  $L_0^p(\Omega, \nu)$ , we obtain an equivariant isomorphism from  $L^p(\Omega, \nu)/L^p(\Omega, \nu)^{\pi_{p,\sigma}}$  to  $L_0^{p'}(\Omega, \nu)'$  given by  $[f] \mapsto \varphi_{f_0}$ , where  $f_0 \in L_0^p(\Omega, \nu)$  is such that  $f = f_0 + c$ , for some  $c \in \mathbb{C}$ . We claim that this map is isometric. Observe that if  $f_0 \in L_0^p(\Omega, \nu)$ ,  $g_0 \in L_0^{p'}(\Omega, \nu)$  and  $c \in \mathbb{C}$  then

$$\int_\Omega f_0(g_0 + c) \, d\nu = \int_\Omega f_0 g_0 \, d\nu = \int_\Omega (f_0 + c)g_0 \, d\nu. \quad (\text{III.5})$$

We apply the first of these equalities to see that

$$\|\varphi_{f_0}\| = \sup \left\{ \left| \int_\Omega f_0 g_0 \, d\nu \right| \mid g \in L_0^{p'}(\Omega, \nu), \|g\|_{p'} \leq 1 \right\} = \|f_0\|_p,$$

from which it follows that

$$\|f_0 + c\| = \inf_{d \in \mathbb{C}} \|f_0 + d\|_p \leq \|f_0\|_p = \|\varphi_{f_0}\|.$$

Conversely, utilizing the second equality in equation III.5, we see that, for each  $d \in \mathbb{C}$ ,

$$\begin{aligned} \|f_0 + d\|_p &= \sup \left\{ \left| \int_{\Omega} (f_0 + d)g \, d\nu \right| \mid g \in L^{p'}(\Omega, \nu), \|g\|_{p'} \leq 1 \right\} \\ &\geq \sup \left\{ \left| \int_{\Omega} f_0 g_0 \, d\nu \right| \mid g_0 \in L_0^{p'}(\Omega, \nu), \|g_0\|_{p'} \leq 1 \right\}. \end{aligned}$$

Hence,

$$\|f_0 + c\| = \inf_{d \in \mathbb{C}} \|f_0 + d\|_p \geq \|\varphi_{f_0}\|.$$

This proves the claim.  $\square$

We shall see in Theorem III.5.5 that, in many cases,  $L_0^p(\Omega, \nu)$  is *not* isometrically isomorphic to an  $L^p$ -space on a  $\sigma$ -finite measure space, and so, neither is its dual. The proof relies on the following extension theorem, which is Theorem 4 in [KK01]. We state it below without proof.

**Theorem III.5.2** (Extension theorem). *Let  $1 \leq p < \infty$ ,  $p \notin 2\mathbb{N}$ , and let  $(\Omega_1, \nu_1)$  and  $(\Omega_2, \nu_2)$  be probability spaces. Let  $Y \subset L^p(\Omega_1, \nu_1)$  be a subspace containing  $1_{\Omega_1}$  and denote by  $\Sigma(Y)$  the smallest  $\sigma$ -algebra on  $\Omega_1$  making all functions in  $Y$  measurable. Let  $\Phi : Y \rightarrow L^p(\Omega_2, \nu_2)$  be a linear isometry. There exists a linear isometry  $\Phi' : L^p(\Omega_1, \Sigma(Y), \nu_1) \rightarrow L^p(\Omega_2, \nu_2)$  such that  $\Phi'|_Y = \Phi$ .*

Before proceeding to the proof of Theorem III.5.5, we shall need two lemmas. Lemma III.5.3 is well-known and can be verified with standard methods. We suspect that Lemma III.5.4 is known, but we do not have a reference.

**Lemma III.5.3.** *Let  $(\Omega, \mu)$  be a  $\sigma$ -finite measure space and let  $1 \leq p < \infty$ . There exists an equivalent probability measure  $\nu$  on  $\Omega$  such that  $L^p(\Omega, \mu)$  is isometrically isomorphic to  $L^p(\Omega, \nu)$ .*

**Lemma III.5.4.** *Let  $(\Omega, \mathcal{B}, \nu)$  be a diffuse standard probability space and let  $1 \leq p < \infty$ . There exists an isometric isomorphism  $U$  of  $L^p(\Omega, \mathcal{B}, \nu)$  such that*

1.  $1_{\Omega} \in U(L_0^p(\Omega, \mathcal{B}, \nu))$ ,
2. *The smallest  $\sigma$ -algebra making all functions in  $U(L_0^p(\Omega, \mathcal{B}, \nu))$  measurable is  $\mathcal{B}$ .*

*Proof.* It suffices to consider the case where  $(\Omega, \mathcal{B}, \nu)$  is the interval  $[0, 1]$  with the Lebesgue measure. Let  $f_0 \in L_0^p([0, 1])$  be such that  $|f_0(\omega)| = 1$ , for all  $\omega \in [0, 1]$ . Then the multiplication operator  $M_{f_0}$  is an isometric isomorphism of  $L^p([0, 1])$ . Because  $L_0^p([0, 1])$  is stable under complex conjugation and  $f_0 \overline{f_0} = 1_{\Omega}$ , we see that  $1_{\Omega} \in M_{f_0}(L_0^p([0, 1]))$ . Further, for each pair  $0 \leq a < b \leq 1$ , set  $f_{a,b} = 1_{(a, (a+b)/2)} - 1_{((a+b)/2, b)}$ . Then  $f_{a,b}$  has mean zero and  $M_{f_0} f_{a,b}$  takes values in  $\mathbb{T}$  on the interval  $(a, b)$  and is zero elsewhere. Therefore, the open interval  $(a, b) = (M_{f_0} f_{a,b})^{-1}(\mathbb{T})$  is in the  $\sigma$ -algebra generated by  $M_{f_0}(L_0^p([0, 1]))$ . Hence,  $M_{f_0}(L_0^p([0, 1]))$  generates the Borel  $\sigma$ -algebra on  $[0, 1]$ .  $\square$

**Theorem III.5.5.** *Let  $(\Omega, \nu)$  be a diffuse standard probability space and let  $1 \leq p < \infty$ ,  $p \notin 2\mathbb{N}$ . Then  $L_0^p(\Omega, \nu)$  is not isometrically isomorphic to an  $L^p$ -space on a  $\sigma$ -finite measure space.*

*Proof.* By Lemma III.5.3, it suffices to show that  $L_0^p(\Omega, \nu)$  is not isometrically isomorphic to an  $L^p$ -space on a probability space. Suppose for contradiction that there is a probability space  $(\Omega', \nu')$  and a linear isometric isomorphism  $\Phi : L_0^p(\Omega, \nu) \rightarrow L^p(\Omega', \nu')$ . Let  $U$  be a linear isometric isomorphism of  $L^p(\Omega, \nu)$  as in Lemma III.5.4. We apply the extension theorem (Theorem III.5.2) to the composition  $\Phi \circ U^{-1} : U(L_0^p(\Omega, \nu)) \rightarrow L^p(\Omega', \nu')$  to obtain a linear isometry  $L^p(\Omega, \nu) \rightarrow L^p(\Omega', \nu')$  extending  $\Phi$ . But  $\Phi \circ U^{-1}$  is already surjective as a map defined on  $U(L_0^p(\Omega, \nu))$ . Therefore, it cannot extend to an injective map out of a strictly larger space. Hence, the map  $\Phi$  cannot exist.  $\square$

The example considered in this section of an ergodic p.m.p. action on a diffuse standard probability space  $(\Omega, \nu)$  is for us the most important example. Indeed, if  $L_0^p(\Omega, \nu)$  is bounded equivariantly isomorphic to an isometric representation on a space in  $L^p$ , the equivalence of weak property  $(T_{L^p})$  and property  $(T)$  would follow from this together with the characterization of property  $(T)$  by Connes and Weiss (see Theorem III.2.4 and Remark III.2.5 in the preliminary section). We end this section by remarking that we have only partially refuted this proof strategy by showing that  $L_0^p(\Omega, \nu)$  is not *isometrically* isomorphic to a space in  $L^p$ . It remains an open question if it is possible to find an equivariant bounded isomorphism. We expect that the answer to this question is ‘No’.

**Question.** Let  $\Gamma \curvearrowright (\Omega, \nu)$  be an ergodic p.m.p. action of a discrete group  $\Gamma$  on a diffuse standard probability space and let  $1 \leq p < \infty$ ,  $p \neq 2$ . Does there exist a  $\sigma$ -finite measure space  $(\Omega', \mu)$  and a bounded (not necessarily isometric) isomorphism  $\Phi : L_0^p(\Omega, \nu) \rightarrow L^p(\Omega', \mu)$  such that  $\Phi \circ \pi_{p,\sigma}^0 \circ \Phi^{-1}$  is an isometric representation of  $\Gamma$  on  $L^p(\Omega', \mu)$ ?

It is known that  $L_0^p(\Omega, \nu)$  is bounded isomorphic to an  $L^p$ -space. This can be shown via the *decomposition method* (see page 14 in [JL01]). However, the bounded isomorphism achieved in this way is not equivariant.

**Acknowledgements** The author is grateful to Mikael de la Salle for the question and for discussions leading to the proof presented in this paper. The author thanks Todor Tsankov for generously sharing his insights into Polish groups. The author thanks Nadia Larsen for comments on an earlier version of this paper leading to improvements in the presentation. The author thanks the anonymous referee for the careful reading of this article and for useful comments. The author thanks the Trond Mohn foundation for supporting a research visit in Lyon where this project was initiated.

## Erratum

1. In Proposition III.5.1, it can also be shown via a standard application of the Hahn-Banach theorem that  $L^p(\Omega, \nu)/L^p(\Omega, \nu)^{\pi_{p,\sigma}}$  is isometrically isomorphic

to the dual of  $L_0^p(\Omega, \nu)$ : For a Banach space  $X$  and a closed subspace  $M \subset X$ ,  $X'/M^\perp$  is isometrically isomorphic to  $M'$ , where  $M^\perp$  denotes the annihilator of  $M$  in  $X$  (see, e.g., Theorem 1.10.16 in [Meg98]).

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## Paper IV

# Symmetrized pseudofunction algebras from $L^p$ -representations and amenability of locally compact groups

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Published in *Expositiones Mathematicae* 43.4 (2025), p. 125685.

**Abstract** We show via an application of techniques from complex interpolation theory how the  $L^p$ -pseudofunction algebras of a locally compact group  $G$  can be understood as sitting between  $L^1(G)$  and  $C^*(G)$ . Motivated by this, we collect and review various characterizations of group amenability connected to the  $p$ -pseudofunction algebra of Herz and generalize these to the symmetrized setting. Along the way, we describe the Banach space dual of the symmetrized pseudofunction algebras on  $G$  associated with representations on reflexive Banach spaces.

## IV.1 Introduction

Let  $G$  be a locally compact group. For  $1 < p < \infty$ , the  $p$ -pseudofunction algebra of  $G$ , which we denote by  $F_{\lambda_p}(G)$ , is the completion of  $L^1(G)$  with respect to the norm associated with the left-regular representation of  $G$  on  $L^p(G)$ . This Banach algebra goes back to the work of Herz from the 1970's (see Section 8 in [Her73]) where it is denoted by  $PF_p(G)$ , and it has been studied intensely in the context of abstract harmonic analysis (see, e.g., [Her76], [CF76], [Der11] and [DS13]). More recently, it has appeared in work by several authors playing the role as an  $L^p$ -analog of the reduced group  $C^*$ -algebra. For example, the simplicity of  $F_{\lambda_p}(G)$  is

studied by Hejazian and Pooya in [PH15] and by Phillips in [Phi19]. Further, Liao and Yu studied the K-theory of  $F_{\lambda_p}(G)$  in [LY17] and Gardella and Thiel showed in [GT22] the strong rigidity result that a locally compact group  $G$  can be recovered from  $F_{\lambda_p}(G)$  when  $1 < p < \infty$  and  $p \neq 2$ . The analogy with the reduced group  $C^*$ -algebra is underlined by Phillips who refers to  $F_{\lambda_p}(G)$  as the *reduced group  $L^p$ -operator algebra of  $G$* . Analogs of the universal group  $C^*$ -algebra have also appeared in the literature in the form of various pseudofunction algebras, where by *pseudofunction algebra* we mean a completion of  $L^1(G)$  with respect to a norm coming from some class of isometric representations of  $G$ . For example, in the work of Gardella and Thiel in [GT14], the role of  $C^*(G)$  is played by the pseudofunction algebra  $F_{L^p}(G)$  associated with the class of isometric representations of  $G$  on  $L^p$ -spaces. The  $L^p$ -pseudofunction algebra  $F_{L^p}(G)$  is also suggested as a natural  $p$ -analog of  $C^*(G)$  by Drutu and Nowak in [DN15]. Another suggestion for a  $p$ -analog of  $C^*(G)$  is the  $QSL^p$ -pseudofunction algebra  $F_{QSL^p}(G)$ , which appears in [Run04] where Runde studies its dual as a  $p$ -analog of the Fourier-Stieltjes algebra.

In this paper, we study symmetrized versions of pseudofunction algebras on  $G$  associated with representations on  $L^p$ - and on  $QSL^p$ -spaces. We are, in particular, interested in the symmetrized  $p$ -pseudofunction algebras  $F_{\lambda_p}^*(G)$  and the symmetrized  $L^p$ -pseudofunction algebras  $F_{L^p}^*(G)$ . The symmetrized  $p$ -pseudofunction algebras were introduced by Liao and Yu in [LY17] in connection with the Baum-Connes conjecture. Later, their simplicity has been studied by Phillips in [Phi19], and they have appeared in the work of Samei and Wiersma in [SW20] and [SW24] where they were studied in connection with quasi-Hermitian groups and exotic group  $C^*$ -algebras, respectively. Unlike their non-symmetrized relatives, the symmetrized pseudofunction algebras are always Banach  $*$ -algebras with the involution coming from  $L^1(G)$ . The starting point for this paper is the commutative diagram below consisting of canonical contractions with dense range:

$$\begin{array}{ccc}
 & L^1(G) & \\
 \swarrow & & \searrow \\
 F_{L^p}^*(G) & \xrightarrow{\quad} & F_{\lambda_p}^*(G) \\
 \swarrow & & \searrow \\
 C^*(G) & \xrightarrow{\quad} & C_r^*(G)
 \end{array}$$

With this diagram in mind, we think of  $F_{\lambda_p}^*(G)$  and  $F_{L^p}^*(G)$  as interpolations between  $L^1(G)$  and the group  $C^*$ -algebras. More precisely, for a pair of Hölder exponents  $1 \leq p < q \leq 2$ , it is shown in Proposition 4.5 in [SW20] that the identity on  $L^1(G)$  extends to a contraction  $F_{\lambda_p}^*(G) \rightarrow F_{\lambda_q}^*(G)$ . This is known to fail in general in the non-symmetrized setting (see Remark 3.19 in [GT14]). In Theorem A, we show the analogous statement for the symmetrized  $L^p$ -pseudofunction algebras via an application of Stein's interpolation theorem. For the non-symmetrized  $L^p$ -pseudofunction algebras, this is shown in Theorem 2.30 in [GT14]. In the non-symmetrized setting, however, the proof is more cumbersome as interpolation techniques are not available.

**Theorem A.** *Let  $G$  be a locally compact group and let  $1 \leq p < q \leq 2$ , The identity map on  $L^1(G)$  extends to a contractive  $*$ -homomorphism  $F_{L^p}^*(G) \rightarrow F_{L^q}^*(G)$  with dense range.*

The notion of amenability goes back to work of von Neumann in [Neu29] and is originally defined in the context of measure theory. Since then, it has proven itself a fundamental concept with equivalent characterizations coming from many different corners of mathematics. We refer to [Run02], [Pie84], Section 2.6 in [BO08] or Appendix G in [BLV08] for introductions to the topic and for an overview of its many connections. In  $C^*$ -algebraic terms, it is the property that the universal and reduced group  $C^*$ -algebras coincide canonically, or, equivalently, that the trivial representation extends to a  $*$ -representation of  $C_r^*(G)$ . With the diagram presented in the previous paragraph in mind, the question of a symmetrized  $L^p$ -generalization naturally arises: Is  $G$  amenable if and only if  $F_{L^p}^*(G)$  and  $F_{\lambda_p}^*(G)$  coincide canonically? We show in Theorem B that the answer to this question is affirmative. This extends Proposition 3.1 of [SW24], which states that  $G$  is amenable if and only if the trivial representation extends to a  $*$ -representation of  $F_{\lambda_p}^*(G)$ . Further, we build upon the work of Cowling in [Cow79] and Runde in [Run04] and give a characterization of amenability in terms of the Banach space dual of the symmetrized  $p$ -pseudofunction algebra,  $F_{\lambda_p}^*(G)'$ , and the  $p$ -Fourier-Stieltjes algebra,  $B_p(G)$ , introduced by Runde in [Run04]. This result is also included in Theorem B.

**Theorem B.** *Let  $G$  be a locally compact group and let  $1 < p, p' < \infty$  be Hölder conjugates. The following are equivalent:*

- (i)  $G$  is amenable,
- (ii)  $F_{\lambda_p}^*(G)'$  is canonically isometrically isomorphic to  $B_p(G) + B_{p'}(G)$ ,
- (iii) The canonical map  $F_{L^p}^*(G) \rightarrow F_{\lambda_p}^*(G)$  is an isometric isomorphism,
- (iv) The canonical map  $F_{L^p}^*(G) \rightarrow F_{\lambda_p}^*(G)$  is an isomorphism,
- (v) The trivial representation  $1_G$  extends to a  $*$ -representation of  $F_{\lambda_p}^*(G)$ .

The equivalence of amenability and the properties (iii) and (iv) of Theorem B should be seen as parallel to Theorem 3.7 in [GT14] where the analogous equivalences are established in the non-symmetrized setting.

To establish the equivalence of amenability and property (ii) of Theorem B, we characterize the Banach space dual of the symmetrized pseudofunction algebra  $F_{\pi}^*(G)$  belonging to a general isometric representation  $\pi$  of  $G$  on a reflexive Banach space, e.g., an  $L^p$ - or  $QSL^p$ -space with  $1 < p < \infty$ . This may be of independent interest.

**Theorem C.** *Let  $(\pi, E)$  be an isometric representation of the locally compact group  $G$  on a reflexive Banach space  $E$ . The canonical identification of  $L^1(G)'$  and  $L^\infty(G)$  restricts to an isometric isomorphism between  $F_{\pi}^*(G)'$  and the sum space  $B_{\pi}(G) + B_{\pi'}(G)$ .*

Here, the norm on the sum space is given, for  $\varphi \in B_\pi(G) + B_{\pi'}(G)$ , by

$$\|\varphi\|_{B_\pi+B_{\pi'}} = \inf \left\{ \|\varphi_0\|_{B_\pi} + \|\varphi_1\|_{B_{\pi'}} \mid \varphi = \varphi_0 + \varphi_1, \varphi_0 \in B_\pi(G), \varphi_1 \in B_{\pi'}(G) \right\}.$$

This paper is organized as follows: In Section IV.2, we recall the  $L^p$ -representation theory for a locally compact group when  $p \neq 2$ . Further, we recall the construction of a symmetrized pseudofunction algebra, and we introduce the tools from complex interpolation theory which we shall need in the paper. In Section IV.3, we prove Theorem A. In Section IV.4, we study the Banach space dual of a pseudofunction algebra and prove Theorem C. Finally, in Section IV.5, we discuss applications to amenability and prove Theorem B.

## IV.2 Preliminaries

**Group actions on  $L^p$ -spaces** Let  $(\Omega, \mu)$  be a  $\sigma$ -finite measure space and let  $1 \leq p < \infty$ ,  $p \neq 2$  be fixed. The group  $\text{Isom}(L^p(\Omega, \mu))$  of surjective isometries of  $L^p(\Omega, \mu)$  is described completely by the Banach-Lamperti theorem, which we recall in Theorem IV.2.1 below. There are two basic types of isometries on  $L^p(\Omega, \mu)$ :

1. We denote by  $L^0(\Omega, \mu; \mathbb{T})$  the collection of measurable functions on  $\Omega$  with values in the unit circle  $\mathbb{T}$  with two functions identified if they differ only on a null set. For each  $c \in L^0(\Omega, \mu; \mathbb{T})$ , the associated *multiplier*  $m_c$  is the surjective isometry on  $L^p(\Omega, \mu)$  given by, for  $\xi \in L^p(\Omega, \mu)$ ,

$$m_c(\xi) = c \cdot \xi.$$

With multiplication defined pointwise,  $L^0(\Omega, \mu; \mathbb{T})$  is a group, and we obtain an injective group homomorphism  $m : L^0(\Omega, \mu; \mathbb{T}) \rightarrow \text{Isom}(L^p(\Omega, \mu))$  by setting  $m(c) = m_c$ .

2. We denote by  $\text{Aut}(\Omega, [\mu])$  the group of all bi-measurable transformations  $\sigma$  of  $\Omega$  that leave  $\mu$  quasi-invariant, i.e., the push forward measure  $\sigma_*\mu$  of  $\mu$  under  $\sigma$  has the same null sets as  $\mu$ . This assumption ensures the existence of the Radon-Nikodym derivative  $\frac{d\sigma_*\mu}{d\mu}$ , which is a real-valued and non-negative function on  $\Omega$ . Define for each  $\xi \in L^p(\Omega, \mu)$ ,

$$u_\sigma \xi = \left( \frac{d\sigma_*\mu}{d\mu} \right)^{1/p} \xi \circ \sigma^{-1}.$$

Then  $u_\sigma$  is a surjective isometry on  $L^p(X, \mu)$ . We obtain an injective map  $u : \text{Aut}(\Omega, [\mu]) \rightarrow \text{Isom}(L^p(\Omega, \mu))$  by setting  $u(\sigma) = u_\sigma$ . It follows from the uniqueness part of the Radon-Nikodym theorem that this is a group homomorphism.

For each pair  $c \in L^0(\Omega, \mu; \mathbb{T})$  and  $\sigma \in \text{Aut}(\Omega, [\mu])$ , a straight forward computation verifies that they satisfy the covariance relation  $u_\sigma m_c u_\sigma^{-1} = m_{c \circ \sigma^{-1}}$ . Thus, we have an injective group homomorphism

$$\begin{array}{ccc} L^0(\Omega, \mu; \mathbb{T}) \rtimes \text{Aut}(\Omega, [\mu]) & \longrightarrow & \text{Isom}(L^p(\Omega, \mu)) \\ (c, \sigma) & \longmapsto & m_c u_\sigma \end{array}$$

The content of the Banach-Lamperti theorem is that this map is surjective when  $p \neq 2$ . This was proven by Lamperti in [Lam58] and prior to that claimed without proof in the monograph [Ban32] by Banach in the special case of the interval equipped with the Lebesgue measure. A modern account in the general setting of Boolean algebras can be found in [Gar21].

**Theorem IV.2.1** (Banach-Lamperti). *Let  $(\Omega, \mu)$  be a  $\sigma$ -finite measure space, let  $1 \leq p < \infty$ ,  $p \neq 2$ , and let  $T$  be a surjective isometry on  $L^p(\Omega, \mu)$ . There exist unique  $c \in L^0(\Omega, \mu; \mathbb{T})$  and  $\sigma \in \text{Aut}(\Omega, [\mu])$  such that  $T = m_c u_\sigma$ .*

Let  $G$  be a locally compact group and let  $(\Omega, \mu)$  be a  $\sigma$ -finite measure space. An isometric representation of  $G$  on  $L^p(\Omega, \mu)$  is a strongly continuous group homomorphism  $G \rightarrow \text{Isom}(L^p(\Omega, \mu))$ . As a corollary to the Banach-Lamperti theorem, we get a complete description of the isometric representations of  $G$  on  $L^p(\Omega, \mu)$  when  $p \neq 2$ . Before stating this in Corollary IV.2.2, we shall need to recall the definition of a measure class preserving action and a 1-cocycle for such an action.

A measure class preserving action of  $G$  on a  $\sigma$ -finite measure space  $(\Omega, \mu)$  is a group homomorphism  $\sigma : G \rightarrow \text{Aut}(\Omega, [\mu])$ . We write  $G \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  for the action given by  $\sigma$ . We shall often omit  $\sigma$  from the notation and write  $t.\omega$  rather than  $\sigma_t(\omega)$ , for  $t \in G$  and  $\omega \in \Omega$ . A ( $\mathbb{T}$ -valued) 1-cocycle for the action  $G \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  is a map  $c : G \times \Omega \rightarrow \mathbb{T}$  such that  $c_t : \Omega \rightarrow \mathbb{T}$  is a measurable map, for every  $t \in G$ , and which satisfies the 1-cocycle relation  $c_{st} = c_s \cdot (c_t \circ \sigma_s^{-1})$   $\nu$ -a.e., for every pair  $s, t \in G$ . The set of all 1-cocycles for  $\sigma$  is denoted by  $Z^1(\sigma; \mathbb{T})$ . Given a measure class preserving action  $G \overset{\sigma}{\curvearrowright} (\Omega, \nu)$  and a 1-cocycle  $c \in Z^1(\sigma; \mathbb{T})$ , we construct an isometric representation of  $G$  on  $L^p(\Omega, \mu)$  as follows: For  $t \in G$ , set  $\pi_{p,\sigma,c}(t) = m_{c_t} u_{\sigma_t}$ . That is, for  $\xi \in L^p(\Omega, \mu)$  and  $\omega \in \Omega$ ,

$$\pi_{p,\sigma,c}(t)\xi(\omega) = c_t(\omega) \left( \frac{ds.\mu}{d\mu} \right)^{1/p} (\omega)\xi(s^{-1}.\omega). \quad (\text{IV.1})$$

**Corollary IV.2.2.** *Let  $G \overset{\sigma}{\curvearrowright} (\Omega, \mu)$  be a measure class preserving action of a locally compact group on a  $\sigma$ -finite measure space, let  $c \in Z^1(\sigma; \mathbb{T})$  and let  $1 \leq p < \infty$ . Then  $\pi_{p,\sigma,c}$  defined in equation (IV.1) is an isometric representation of  $G$  on  $L^p(\Omega, \mu)$ . Moreover, if  $p \neq 2$ , all isometric representations on  $L^p(\Omega, \mu)$  have this form.*

**Symmetrized pseudofunction algebras** We recall in the following the construction of a symmetrized pseudofunction algebra. We refer to [DN15], [EP24] and [GT14] for a more thorough treatment of general pseudofunction algebras.

Let  $G$  be a locally compact group. Given an isometric representation  $\pi$  of  $G$  on a Banach space  $E$ , its integrated form is the contractive, non-degenerate Banach

algebra representation of  $L^1(G)$  on  $E$  given, for  $f \in L^1(G)$ , by

$$\pi(f) = \int_G f(s)\pi(s) d\mu_G(s),$$

where  $\mu_G$  denotes the Haar-measure on  $G$ . It is folklore that integration gives a 1-to-1 correspondence between the isometric representations of  $G$  and the contractive, non-degenerate representations of  $L^1(G)$ . We denote by  $E'$  the Banach space dual of  $E$ . The isometric representation  $(\pi, E)$  gives, in a natural way, rise to an isometric representation on  $E'$  as follows: For  $t \in G$ ,  $\xi \in E$  and  $\eta \in E'$ , set

$$[\pi'(t)\eta](\xi) = \eta(\pi(t^{-1})\xi).$$

We refer to  $(\pi', E')$  as the *dual representation* of  $(\pi, E)$ . We assume in the following that  $E$  is reflexive so that  $\pi''$  can be identified with  $\pi$ . We associate to  $\pi$  a seminorm on  $L^1(G)$  as follows: For  $f \in L^1(G)$ , set

$$\|f\|_{F_\pi^*} = \max\{\|\pi(f)\|, \|\pi'(f)\|\}.$$

This defines a norm on the quotient of  $L^1(G)$  with  $\ker \pi \cap \ker \pi'$ . The completion with respect to this norm is denoted by  $F_\pi^*(G)$  and referred to as the *symmetrized  $\pi$ -pseudofunction algebra* of  $G$ . In this paper, we are, in particular, interested in the case where  $\pi$  is the left-regular representation  $\lambda_p$  of  $G$  on  $L^p(G)$ , for  $1 < p < \infty$ . The symmetrized pseudofunction algebra  $F_{\lambda_p}^*(G)$  associated with  $\lambda_p$  is referred to as the *symmetrized  $p$ -pseudofunction algebra* of  $G$ . It is shown in Proposition 4.2 in [SW20] that  $F_{\lambda_p}^*(G)$  is a Banach \*-algebra.

Let  $\mathcal{E}$  be a class of reflexive Banach space and let  $\mathcal{E}'$  be the class of Banach spaces which are dual to the spaces in  $\mathcal{E}$ . We denote by  $\text{Rep}_\mathcal{E}(G)$  the class of isometric representations of  $G$  on spaces in  $\mathcal{E}$ . We associate to  $\mathcal{E}$  a seminorm on  $L^1(G)$  as follows: For  $f \in L^1(G)$ , set

$$\|f\|_{F_\mathcal{E}^*} = \sup\{\|\pi(f)\| \mid \pi \in \text{Rep}_\mathcal{E}(G) \text{ or } \pi \in \text{Rep}_{\mathcal{E}'}(G)\}.$$

Set  $I_\mathcal{E} = \bigcap_{\pi \in \text{Rep}_\mathcal{E}(G)} \ker \pi$ . The seminorm above defines a norm on the quotient of  $L^1(G)$  with  $I_\mathcal{E} \cap I_{\mathcal{E}'}$ . We denote by  $F_\mathcal{E}^*(G)$  the completion with respect to this norm and refer to it as the *symmetrized  $\mathcal{E}$ -pseudofunction algebra*. In the cases we consider, e.g., the classes  $L^p$ , respectively,  $QSL^p$ , it is a Banach \*-algebra with involution coming from  $L^1(G)$ . The proof of this is analogous to the proof of Proposition 4.2 in [SW20] with the obvious adjustments. Further,  $F_\mathcal{E}^*(G)$  is  $\mathcal{E}$ -universal in the sense that, for any  $\pi \in \text{Rep}_\mathcal{E}(G)$ , the identity map on  $L^1(G)$  extends to a contraction  $F_\mathcal{E}^*(G) \rightarrow F_\pi^*(G)$ .

**Spaces of matrix coefficients** Let  $(\pi, E)$  be an isometric representation of the locally compact group  $G$  on a Banach space  $E$ . The  $\pi$ -Fourier-Stieltjes space is the linear subspace of  $L^\infty(G)$  given by

$$B_\pi(G) = \left\{ \varphi : G \rightarrow \mathbb{C} \text{ measurable} \mid \exists C > 0 : |\varphi(f)| \leq C \|\pi(f)\|, \forall f \in L^1(G) \right\},$$

where

$$\varphi(f) = \int_G f(s)\varphi(s) d\mu_G(s).$$

We equip  $B_\pi(G)$  with the norm

$$\|\varphi\|_{B_\pi} = \inf \left\{ C > 0 \mid |\varphi(f)| \leq C \|\pi(f)\|, \forall f \in L^1(G) \right\}.$$

With this norm, the canonical embedding of  $B_\pi(G)$  into  $L^\infty(G)$  is a contraction.

A *matrix coefficient* of the representation  $(\pi, E)$  is a function on  $G$  of the form

$$\varphi_{\xi, \eta}(t) = \langle \pi(t)\xi, \eta \rangle,$$

for  $t \in G$ , where  $\xi \in E$ ,  $\eta \in E'$  and  $\langle \cdot, \cdot \rangle$  is the duality pairing between  $E$  and  $E'$ . Clearly,  $\varphi_{\xi, \eta}$  is an element of  $B_\pi(G)$  with  $\|\varphi_{\xi, \eta}\|_{B_\pi} \leq \|\xi\| \|\eta\|$ . In general, not all elements of  $B_\pi(G)$  need to be matrix coefficients of  $\pi$ , but the  $\pi$ -Fourier-Stieltjes space can still be understood very concretely as a space of matrix coefficients. This is made precise in Theorem IV.2.3 below, which is Theorem 2 in [CF84].

**Theorem IV.2.3.** *Let  $\pi$  be an isometric representation of  $G$  on a Banach space  $E$ . There exists an isometric representation  $\pi_0$  on a Banach space  $E_0$  such that the identity on  $L^\infty(G)$  restricts to an isometric isomorphism between  $B_\pi(G)$  and  $B_{\pi_0}(G)$ , and such that, for every  $\varphi \in B_\pi(G)$ , one can find  $\xi \in E_0$  and  $\eta \in E'_0$  such that  $\varphi = \langle \pi_0(\square)\xi, \eta \rangle$  and  $\|\varphi\|_{B_\pi} = \|\xi\| \|\eta\|$ . Moreover, if  $E$  is an  $L^p$ -space, or a  $QSL^p$ -space, then so is  $E_0$ .*

A  $p$ -analogue of the Fourier-Stieltjes algebra was proposed by Runde in [Run04]. Denote by  $QSL^p$  the class of all Banach spaces isometrically isomorphic to a quotient of a subspace of an  $L^p$ -space. Further, we denote by  $\text{Rep}_p(G)$  the class of isometric representations of  $G$  on a space in  $QSL^p$ . The  $p$ -Fourier-Stieltjes algebra is the set of matrix coefficients of representations in  $\text{Rep}_p(G)$ :

$$B_p(G) = \left\{ \langle \pi(\square)\xi, \eta \rangle \mid (\pi, E) \in \text{Rep}_p(G), \xi \in E, \eta \in E' \right\}.$$

We equip  $B_p(G)$  with the following norm: For  $\varphi \in B_p(G)$ , set

$$\|\varphi\|_{B_p} = \inf \left\{ \|\xi\| \|\eta\| \mid \varphi = \langle \pi(\square)\xi, \eta \rangle, \text{ for } (\pi, E) \in \text{Rep}_p(G), \xi \in E, \eta \in E' \right\}.$$

It is shown in [Run04] that  $B_p(G)$  is a commutative Banach algebra over  $\mathbb{C}$  with pointwise operations. Clearly, it embeds canonically contractively into  $L^\infty(G)$ .

**Warning!** In [Run04], Runde defines  $B_p(G)$  as the set of matrix coefficients of representations in  $\text{Rep}_{p'}(G)$ , where  $p'$  is the Hölder conjugate of  $p$ . We follow the convention used in [Daw10] and do not exchange  $p$  and  $p'$ . Hence,  $B_p(G)$  in our notation is  $B_{p'}(G)$  in the notation of Runde.

Let  $(\pi, E)$  and  $(\rho, F)$  be two isometric Banach space representations of  $G$ . We say that  $\rho$  is *contained* in  $\pi$  and write  $\rho \leq \pi$  if there exists a linear isometry  $T : F \rightarrow E$  such that  $\pi(t)T\xi = T\rho(t)\xi$ , for all  $t \in G$  and  $\xi \in F$ . An isometric representation  $(\pi, E)$  is said to be *cyclic* if there exists an element  $\xi \in E$  such that  $\pi(L^1(G))\xi$  is dense in  $E$ . We denote by  $\text{Cyc}_p(G)$  the subclass of  $\text{Rep}_p(G)$  consisting of all isometric representations on a  $QSL^p$ -space that are cyclic. Unlike  $\text{Rep}_p(G)$ ,  $\text{Cyc}_p(G)$  is a set (see the second remark following Definition 4.2 in [Run04]). An isometric representation  $\pi \in \text{Rep}_p(G)$  is said to be  $p$ -*universal* if it contains all representations from  $\text{Cyc}_p(G)$ . Such a representation exists by the example following Definition 4.5 in [Run04].

**Theorem IV.2.4.** *Let  $(\pi, E)$  be an isometric  $QSL^p$ -representation of the locally compact group  $G$ . Then  $B_\pi(G)$  embeds canonically and contractively into  $B_p(G)$ . If  $(\pi, E)$  is  $p$ -universal, this embedding is an isometric isomorphism.*

*Proof.* It is a direct consequence of Theorem IV.2.3 that  $B_\pi(G)$  embeds contractively into  $B_p(G)$ . Indeed, let  $\varphi \in B_\pi(G)$ . By Theorem IV.2.3 we may find a  $QSL^p$ -representation  $(\pi_0, E_0)$  and elements  $\xi \in E$  and  $\eta \in E'$  such that  $\varphi = \langle \pi(\square)\xi, \eta \rangle$  and  $\|\varphi\|_{B_\pi} = \|\xi\| \|\eta\|$ . Then  $\varphi$  lies in  $B_p(G)$  with  $\|\varphi\|_{B_p} \leq \|\varphi\|_{B_\pi}$ . Suppose now that  $(\pi, E)$  is  $p$ -universal and let  $\psi \in B_p(G)$ . By definition of  $B_p(G)$  we may find a  $QSL^p$ -representation  $(\pi_1, E_1)$  such that  $\psi = \langle \pi_1(\square)\xi, \eta \rangle$ , for some  $\xi \in E_1$  and  $\eta \in E'_1$ . By the remark following Definition 4.1 in [Run04], we may take  $\pi_1$  to be cyclic. Because  $\pi$  is  $p$ -universal, there is a linear isometry  $T : E_1 \rightarrow E$  such that  $\pi(t)T\xi = T\pi_1(t)\xi$ , for all  $\xi \in E_1$  and  $t \in G$ . Then, for any  $f \in L^1(G)$  and  $\zeta \in E_1$ ,

$$\|\pi_1(f)\zeta\| = \|T\pi_1(f)\zeta\| = \|\pi(f)T\zeta\| \leq \|\pi(f)\| \|\zeta\|.$$

It follows that  $\|\pi_1(f)\| \leq \|\pi(f)\|$ , and so,  $|\psi(f)| = |\langle \pi_1(f)\xi, \eta \rangle| \leq \|\pi(f)\| \|\xi\| \|\eta\|$ , for every  $f \in L^1(G)$ . Hence,  $\psi$  lies in  $B_\pi(G)$  with  $\|\psi\|_{B_\pi} \leq \|\xi\| \|\eta\|$ . Since  $\psi = \langle \pi_1(\square)\xi, \eta \rangle$  was an arbitrary representation of  $\psi$  we can take the infimum on the right hand side of this inequality to obtain  $\|\psi\|_{B_\pi} \leq \|\psi\|_{B_p}$ . Hence, when  $\pi$  is  $p$ -universal, the canonical embedding  $B_\pi(G) \hookrightarrow B_p(G)$  is an isometric isomorphism.  $\square$

**Complex interpolation** We give a brief overview of the complex interpolation method focusing on interpolation bounds on families of operators. We refer the reader to [BL11] for a thorough introduction to the topic. A pair  $(E_0, E_1)$  of complex Banach spaces is said to be *compatible* if there exists a Hausdorff topological vector space  $V$  and  $\mathbb{C}$ -linear continuous embeddings  $\iota_j : E_j \hookrightarrow V$ , for  $i \in \{0, 1\}$ . Given a compatible pair  $(E_0, E_1)$ , their *intersection space* is the vector subspace of  $V$  given by  $E_0 \cap E_1 = \iota_0(E_0) \cap \iota_1(E_1)$ . It becomes a Banach space when equipped with the norm defined for  $\xi = \iota_0(\xi_0) = \iota_1(\xi_1)$  by

$$\|\xi\|_{E_0 \cap E_1} = \max\{\|\xi_0\|_{E_0}, \|\xi_1\|_{E_1}\},$$

Further, the *sum space* of the pair  $(E_0, E_1)$  is the vector subspace of  $V$  given by  $E_0 + E_1 = \iota_0(E_0) + \iota_1(E_1)$ . We equip this space with the Banach space norm

$$\|\xi\|_{E_0 + E_1} = \inf \left\{ \|\xi_0\|_{E_0} + \|\xi_1\|_{E_1} \mid \xi = \iota_0(\xi_0) + \iota_1(\xi_1) \right\}.$$

The complex interpolation method associates to each parameter  $\theta \in [0, 1]$  a Banach space  $[E_0, E_1]_\theta$  such that there are canonical continuous inclusions  $E_0 \cap E_1 \subset [E_0, E_1]_\theta \subset E_0 + E_1$ , and such that  $E_0 \cap E_1$  is dense in  $[E_0, E_1]_\theta$ . For a measure space  $(\Omega, \mu)$  and parameters  $1 < p_0 < p_1 < \infty$ , the interpolation space of  $L^{p_0}(\Omega, \mu)$  and  $L^{p_1}(\Omega, \mu)$  with parameter  $\theta \in [0, 1]$  can be identified with  $L^{p_\theta}(\Omega, \mu)$  with

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Viewed as a function on  $\theta \in [0, 1]$ ,  $p_\theta$  is continuous and monotonically increasing taking the value  $p_0$  at  $\theta = 0$  and  $p_1$  at  $\theta = 1$ . Denote by  $S(\Omega)$  and  $L^0(\Omega, \mu)$  the simple, respectively, measurable complex valued functions on  $\Omega$ . Given a linear operator  $S(\Omega) \rightarrow L^0(\Omega, \mu)$  which extends to a bounded operator on both  $L^{p_0}(\Omega, \mu)$  and  $L^{p_1}(\Omega, \mu)$ , the Riesz-Thorin theorem ensures that it also extends to a bounded operator on the interpolation spaces. The Riesz-Thorin theorem was generalized by Stein in [Ste56] to families of linear operators. We recall Stein's interpolation theorem in Theorem IV.2.5 below. Denote by

$$\mathbb{S} = \{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z \leq 1\} = \{\theta + i\gamma \mid 0 \leq \theta \leq 1, \gamma \in \mathbb{R}\}$$

the vertical *strip* in the complex plane. A function  $\Phi : \mathbb{S} \rightarrow \mathbb{C}$  which is continuous on  $\mathbb{S}$  and analytic on the interior  $\mathbb{S}^\circ$  is said to have *admissible growth* if there exists a constant  $k < \pi$  such that

$$\sup_{z \in \mathbb{S}} e^{-k|\operatorname{Im} z|} \log |\Phi(z)| < \infty.$$

A family  $(T_z)_{z \in \mathbb{S}}$  of linear operators  $S(\Omega) \rightarrow L^0(\Omega, \mu)$  indexed by the strip is said to be *admissible* if, for every pair of simple functions  $f, g \in S(\Omega)$ , the map  $\mathbb{S} \rightarrow \mathbb{C}$  given by

$$z \mapsto \int_{\Omega} (T_z f) g \, d\mu$$

is continuous on  $\mathbb{S}$ , analytic on  $\mathbb{S}^\circ$ , and has admissible growth. Given a family  $(T_z)_{z \in \mathbb{S}}$  of admissible operators such that the operators on the left boundary of the strip extend to bounded operators on  $L^{p_0}(\Omega, \mu)$  and the operators on the right boundary of the strip extend to bounded operators on  $L^{p_1}(\Omega, \mu)$ , Stein's interpolation theorem ensures that, for each interpolation parameter  $\theta \in [0, 1]$ , the operator  $T_\theta$  extends to a bounded operator on the interpolation space  $L^{p_\theta}(\Omega, \mu)$ .

**Theorem IV.2.5** (Stein's interpolation theorem). *Let  $1 \leq p_0 < p_1 \leq \infty$  and let  $(\Omega, \mu)$  be a measure space. Suppose  $(T_z)_{z \in \mathbb{S}}$  is an admissible family of linear operators  $S(\Omega) \rightarrow L^0(\Omega, \mu)$  satisfying*

$$\|T_{i\gamma}\xi\|_{p_0} \leq M_0(\gamma) \|\xi\|_{p_0} \quad \text{and} \quad \|T_{i\gamma+1}\xi\|_{p_1} \leq M_1(\gamma) \|\xi\|_{p_1},$$

for all simple functions  $\xi \in S(\Omega)$  and all  $\gamma \in \mathbb{R}$ , where  $M_j(\gamma) > 0$ , for  $j \in \{0, 1\}$ , are independent of  $\xi$  and satisfy

$$M_j \doteq \sup_{\gamma \in \mathbb{R}} \log M_j(\gamma) < \infty.$$

Then, for each  $0 < \theta < 1$ , the constant  $M_\theta > 0$  defined by

$$\log M_\theta = \frac{\sin \pi\theta}{2} \int_{-\infty}^{\infty} \frac{M_0}{\cosh \pi\gamma - \cos \pi\theta} + \frac{M_1}{\cosh \pi\gamma + \cos \pi\theta} \, d\gamma$$

is finite, and

$$\|T_\theta \xi\|_{p_\theta} \leq M_\theta \|\xi\|_{p_\theta},$$

for every simple function  $\xi \in S(\Omega)$ .

### IV.3 Interpolations of $L^1(G)$ and $C^*(G)$

In this section we prove Theorem A from the introduction, which is Theorem IV.3.3 below. This establishes canonical contractive  $*$ -homomorphisms  $F_{L^p}^*(G) \rightarrow F_{L^q}^*(G)$ , for  $1 \leq p < q \leq 2$ . The proof relies on Stein's interpolation theorem and on the Banach-Lamperti theorem. When  $q = 2$  the Banach-Lamperti theorem does not apply. However, this obstacle can be circumvented with an application of the so-called "Gaussian functor trick".

Let  $G \overset{\sigma}{\curvearrowright} (\Omega, \mu)$  be a measure class preserving action of the locally compact group  $G$  on the  $\sigma$ -finite measure space  $(\Omega, \mu)$ . Fix  $1 \leq p_0 < p_1 < \infty$ . For each  $s \in G$  and  $\gamma \in \mathbb{R}$ , define  $a_s^\gamma : \Omega \rightarrow \mathbb{C}$  by

$$a_s^\gamma = \left( \frac{ds \cdot \mu}{d\mu} \right)^{\gamma(1/p_0 - 1/p_1)i}. \quad (\text{IV.2})$$

**Lemma IV.3.1.** *The map  $a^\gamma : G \times \Omega \rightarrow \mathbb{T}$  given by  $(s, \omega) \mapsto a_s^\gamma(\omega)$  with  $a_s^\gamma$  as in equation (IV.2) is a  $\mathbb{T}$ -valued 1-cocycle for the action  $G \overset{\sigma}{\curvearrowright} (\Omega, \mu)$ .*

*Proof.* Recall that the Radon-Nikodym derivative is real-valued and measurable function. Hence,  $a_s^\gamma \in L^0(\Omega, \mu; \mathbb{T})$ , for all  $s \in G$ . Further, for each pair  $s, t \in G$ , the Radon-Nikodym derivative satisfies the following equality:

$$\frac{d(st) \cdot \mu}{d\mu} = \frac{ds \cdot (t \cdot \mu)}{d\mu} = \frac{ds \cdot (t \cdot \mu)}{ds \cdot \mu} \frac{ds \cdot \mu}{d\mu}.$$

Taking both sides to the power  $\gamma(1/p_0 - 1/p_1)i$ , yields the 1-cocycle relation:

$$a_{st}^\gamma = (a_t^\gamma \circ \sigma_s^{-1}) a_s^\gamma. \quad \square$$

Let  $c$  be a  $\mathbb{T}$ -valued 1-cocycle for the action  $G \overset{\sigma}{\curvearrowright} (\Omega, \mu)$ . Recall that set of 1-cocycles is a group with multiplication given entrywise. Hence,  $ca^\gamma$  is again a 1-cocycle, for each  $\gamma \in \mathbb{R}$ , with  $a^\gamma$  as in Lemma IV.3.1.

We denote by

$$\mathbb{S} = \{ z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z \leq 1 \} = \{ \theta + i\gamma \mid 0 \leq \theta \leq 1, \gamma \in \mathbb{R} \}$$

the vertical *strip* in the complex plane. To each  $z = \theta + i\gamma \in \mathbb{S}$  we associate the 1-cocycle  $ca^\gamma$  and the Hölder exponent  $p_0 \leq p_\theta \leq p_1$  which is the unique real number such that  $1/p_\theta = \theta/p_0 + (1 - \theta)/p_1$ . Fix  $f \in L^1(G)$ . For each  $z = \theta + i\gamma \in \mathbb{S}$ , we define a linear operator  $T_{\theta+i\gamma} : S(\Omega) \rightarrow L^0(\Omega, \mu)$  by, for  $\xi \in S(\Omega)$  and  $\omega \in \Omega$ ,

$$T_{\theta+i\gamma}\xi(\omega) = \int_G f(s) (c_s a_s^\gamma)(\omega) \left( \frac{ds \cdot \mu}{d\mu} \right)^{1/p_\theta}(\omega) \xi(s^{-1} \cdot \omega) d\mu_G(s). \quad (\text{IV.3})$$

That is,  $T_{\theta+i\gamma} = \pi_{p_\theta, \sigma, ca^\gamma}(f)$ . In particular, we see that  $T_\theta = \pi_{p_\theta, \sigma, c}(f)$ ,  $T_{i\gamma} = \pi_{p_0, \sigma, ca^\gamma}(f)$  and  $T_{1+i\gamma} = \pi_{p_1, \sigma, ca^\gamma}(f)$ . Lemma IV.3.2 establishes that the family  $(T_z)_{z \in \mathbb{S}}$  is *admissible*.

**Lemma IV.3.2.** *Let  $(T_z)_{z \in \mathbb{S}}$  be the family of bounded linear operators  $S(\Omega) \rightarrow L^0(\Omega, \mu)$  defined in (IV.3). For every pair of simple functions  $\xi, \eta \in S(\Omega)$ , the map  $\mathbb{S} \rightarrow \mathbb{C}$  given by*

$$z \mapsto \int_{\Omega} (T_z \xi) \eta \, d\mu \quad (\text{IV.4})$$

is continuous on  $\mathbb{S}$  and analytic on  $\mathbb{S}^\circ$ . Moreover,

$$\sup_{z \in \mathbb{S}} \log \left| \int_{\Omega} (T_z \xi) \eta \, d\mu \right| < \infty. \quad (\text{IV.5})$$

*Proof.* Fix  $\xi, \eta \in S(\Omega)$ . The map in (IV.4) is continuous on  $\mathbb{S}$  and analytic on  $\mathbb{S}^\circ$  if and only if the two maps

$$\theta \mapsto \int_{\Omega} (T_{\theta+i\gamma} \xi) \eta \, d\mu \quad \text{and} \quad \gamma \mapsto \int_{\Omega} (T_{\theta+i\gamma} \xi) \eta \, d\mu \quad (\text{IV.6})$$

are continuous on  $[0, 1]$  and on  $\mathbb{R}$ , respectively, and  $C^\infty$  on  $(0, 1)$  and on  $\mathbb{R}$ , respectively.

By the Tonelli-Fubini Theorem, we have

$$\begin{aligned} \int_{\Omega} (T_{\theta+i\gamma} \xi) \eta \, d\mu &= \int_{\Omega} \eta \int_G f(s) \pi_{p_\theta, \sigma, ca^\gamma}(s) \xi \, d\mu_G(s) \, d\mu \\ &= \int_G f(s) \int_{\Omega} [\pi_{p_\theta, \sigma, ca^\gamma}(s) \xi] \eta \, d\mu \, d\mu_G(s). \end{aligned}$$

Consider, for each  $s \in G$ , the integrand of the inner integral as a function in the three variables  $(\theta, \gamma, \omega) \in [0, 1] \times \mathbb{R} \times \Omega$ :

$$\begin{aligned} (\theta, \gamma, \omega) &\mapsto [\pi_{p_\theta, \sigma, ca^\gamma}(s) \xi](\omega) \eta(\omega) \\ &= (c_s a_s^\gamma)(\omega) \left( \frac{ds \cdot \mu}{d\mu} \right)^{1/p_\theta} (\omega) \xi(s^{-1} \cdot \omega) \eta(\omega). \end{aligned} \quad (\text{IV.7})$$

This function is measurable in  $\omega$  and continuous in  $\theta$  and  $\gamma$ , respectively. Further, since  $c_s a_s^\gamma$  takes values in  $\mathbb{T}$ , and since  $x^{1/p}$  is monotone as a function in  $p$ , we have

$$\begin{aligned} &|[\pi_{p_\theta, \sigma, ca^\gamma}(s) \xi](\omega) \eta(\omega)| \\ &\leq \max \left\{ \left| \left( \frac{ds \cdot \mu}{d\mu}(\omega) \right)^{1/p_0} \xi(s^{-1} \cdot \omega) \eta(\omega) \right|, \left| \left( \frac{ds \cdot \mu}{d\mu}(\omega) \right)^{1/p_1} \xi(s^{-1} \cdot \omega) \eta(\omega) \right| \right\}. \end{aligned}$$

An application of Hölder's inequality and the change of variables formula shows that both terms in this maximum are integrable functions with integrals bounded by  $\|\xi\|_{p_0} \|\eta\|_{p'_0}$  and  $\|\xi\|_{p_1} \|\eta\|_{p'_1}$ , respectively. Lebesgue's dominated convergence theorem then implies continuity in each of the coordinates of the map  $[0, 1] \times \mathbb{R} \rightarrow \mathbb{C}$  given by

$$(\theta, \gamma) \mapsto \int_{\Omega} [\pi_{p_\theta, \sigma, ca^\gamma}(s) \xi] \eta \, d\mu. \quad (\text{IV.8})$$

A second application of the Lebesgue dominated convergence theorem implies continuity of the maps in equation (IV.6). The argument that the two maps

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in (IV.6) are differentiable on  $(0, 1)$  and on  $\mathbb{R}$ , respectively, is analogous. Observe that  $a^\gamma$  is differentiable in  $\gamma$  and that  $(ds.\mu/d\mu)^{1/p_\theta}$  is differentiable in  $\theta$ . We have,

$$\begin{aligned}\frac{\partial}{\partial \gamma} [\pi_{p_\theta, \sigma, ca^\gamma} \xi](\omega) \eta(\omega) &= \left( \frac{1}{p_0} - \frac{1}{p_1} \right) i \log \left( \frac{ds.\mu}{d\mu}(\omega) \right) [\pi_{p_\theta, \sigma, ca^\gamma} \xi](\omega) \eta(\omega), \\ \frac{\partial}{\partial \theta} [\pi_{p_\theta, \sigma, ca^\gamma} \xi](\omega) \eta(\omega) &= \left( \frac{1}{p_0} - \frac{1}{p_1} \right) \log \left( \frac{ds.\mu}{d\mu}(\omega) \right) [\pi_{p_\theta, \sigma, ca^\gamma} \xi](\omega) \eta(\omega).\end{aligned}$$

The function  $\log(ds.\mu/d\mu)\eta$  lies in  $L^p(\Omega, \mu)$ , for any  $1 < p < \infty$ , because the measure of the support of  $\eta$  is finite. Hence, both partial derivatives are bounded by an integrable function not depending on the parameters  $\theta$  and  $\gamma$ . An application of the Lebesgue dominated convergence theorem then implies differentiability of the map of equation (IV.8) in  $\theta$  on  $(0, 1)$  and in  $\gamma$  on  $\mathbb{R}$ . A second application of the Lebesgue dominated convergence theorem implies differentiability of the maps in equation (IV.6).

It remains to show the inequality of equation (IV.5). With  $p'_\theta$  denoting the Hölder conjugate of  $p_\theta$ , Hölder's inequality yields that

$$\left| \int_{\Omega} (T_{\theta+i\gamma} \xi) \eta \, d\mu \right| \leq \|\pi_{q_\theta, \sigma, ca^\gamma}(f)\| \|\xi\|_{p_\theta} \|\eta\|_{p'_\theta} \leq \|f\|_1 \|\xi\|_{p_\theta} \|\eta\|_{p'_\theta}.$$

The right hand side of this inequality depends only on  $\theta$  and not on  $\gamma$ . Hence,

$$\sup_{z \in \mathbb{S}} \left| \int_{\Omega} (T_z \xi) \eta \, d\mu \right| \leq \|f\|_1 \sup_{\theta \in [0,1]} \|\xi\|_{p_\theta} \|\eta\|_{p'_\theta} < \infty.$$

The inequality of equation (IV.5) follows.  $\square$

With an admissible family at hand, we may employ Stein's interpolation theorem.

**Theorem IV.3.3.** *Let  $G$  be a locally compact second countable group and let  $1 \leq p < q \leq 2$ . The identity map on  $L^1(G)$  extends to a contractive \*-homomorphism  $F_{L^p}^*(G) \rightarrow F_{L^q}^*(G)$  with dense range.*

*Proof.* Let  $G \overset{\sigma}{\curvearrowright} (\Omega, \mu)$  be a measure class preserving action, let  $c$  be a 1-cocycle for this action and consider the representation  $\pi_{q, \sigma, c}$  of  $G$  on  $L^q(\Omega, \mu)$ . We denote by  $p'$  the Hölder conjugate of  $p$ . Let  $f \in L^1(G)$ . For each  $z \in \mathbb{S}$ , let  $T_z : S(\Omega, \mu) \rightarrow L^0(\Omega, \mu)$  be the operator defined in (IV.3) with  $p$  in place of  $p_0$  and  $p'$  in place of  $p_1$ . That is, for each  $\theta \in [0, 1]$  and  $\gamma \in \mathbb{R}$ ,  $T_{\theta+i\gamma} = \pi_{p_\theta, \sigma, ca^\gamma}(f)$ , where  $p \leq p_\theta \leq p'$  is the unique real number such that  $1/p_\theta = \theta/p + (1-\theta)/p'$ , and where  $a^\gamma$  be the 1-cocycle defined, for each  $s \in G$ , by

$$a_s^\gamma = \left( \frac{ds.\mu}{d\mu} \right)^{\gamma(1/p-1/p')i}.$$

Then  $(T_z)_{z \in \mathbb{S}}$  is an admissible family by Lemma IV.3.2. Define  $M_j : \mathbb{R} \rightarrow \mathbb{R}_+$ , for  $j \in \{0, 1\}$ , by

$$M_0(\gamma) = \|\pi_{p, \sigma, ca^\gamma}(f)\| \quad \text{and} \quad M_1(\gamma) = \|\pi_{p', \sigma, ca^\gamma}(f)\|.$$

Then  $M_j(\gamma) \leq \|f\|_1$ , for  $j \in \{0, 1\}$  and for all  $\gamma \in \mathbb{R}$ . Because  $p < q < p'$ , we may find a  $0 < \theta_q < 1$  such that  $q = p\theta_q$ . We obtain from Stein's interpolation theorem, and by using that the simple functions are dense in  $L^q(\Omega, \mu)$ , that  $\|\pi_{q,\sigma,c}(f)\|_q \leq M_{\theta_q}$ , where

$$\log M_{\theta_q} = \frac{\sin \pi\theta_q}{2} \int_{-\infty}^{\infty} \frac{M_0}{\cosh \pi\gamma - \cos \pi\theta_q} + \frac{M_1}{\cosh \pi\gamma + \cos \pi\theta_q} d\gamma, \quad (\text{IV.9})$$

and

$$M_0 = \sup_{\gamma \in \mathbb{R}} \log \|\pi_{p,\sigma,c\alpha\gamma}(f)\|, \quad \text{and} \quad M_1 = \sup_{\gamma \in \mathbb{R}} \log \|\pi_{p',\sigma,c\alpha\gamma}(f)\|.$$

The representations appearing in the definitions of  $M_0$ , respectively,  $M_1$ , are all representations on  $L^p(\Omega, \mu)$ , respectively, on  $L^{p'}(\Omega, \mu)$ . Thus, both  $M_0$  and  $M_1$  are upper bounded by  $\log \|f\|_{F_{L^p}^*}$ . We insert this into equation (IV.9) to obtain an upper bound on  $M_{\theta_q}$ :

$$\log M_{\theta_q} \leq \left( \frac{\sin \pi\theta_q}{2} \int_{-\infty}^{\infty} \frac{1}{\cosh \pi\gamma - \cos \pi\theta_q} + \frac{1}{\cosh \pi\gamma + \cos \pi\theta_q} d\gamma \right) \log \|f\|_{F_{L^p}^*}.$$

A computation shows that the number in the parenthesis on the right hand side of this inequality is always equal to 1 independently of  $\theta_q$ :

$$\begin{aligned} & \frac{\sin \pi\theta_q}{2} \int_{-\infty}^{\infty} \frac{1}{\cosh \pi\gamma - \cos \pi\theta_q} + \frac{1}{\cosh \pi\gamma + \cos \pi\theta_q} d\gamma \\ &= \left[ \frac{1}{\pi} \tan^{-1} \left( \frac{\sinh \pi\gamma}{\sin \pi\theta_q} \right) \right]_{-\infty}^{\infty} = 1. \end{aligned}$$

Hence,

$$\|\pi_{q,\sigma,c}(f)\| \leq M_{\theta_q} \leq \|f\|_{F_{L^p}^*}. \quad (\text{IV.10})$$

The same argument also works with the Hölder conjugate  $q'$  in place of  $q$ .

We have two cases to consider:  $q < 2$  and  $q = 2$ . Suppose first that  $q < 2$ . Then, by the Banach-Lamperti theorem, all isometric representations on an  $L^q$ -space have the form  $\pi_{q,\sigma,c}$ . Since we have  $q' > 2$ , the isometric representations on an  $L^{q'}$ -space also have this form. Thus, taking the supremum over all representations of  $G$  on any  $L^q$ - or  $L^{q'}$ -space, we see that  $\|f\|_{F_{L^q}^*} \leq \|f\|_{F_{L^p}^*}$ . As  $f \in L^1(G)$  was arbitrary, the claim follows.

Suppose now that  $q = 2$  so that  $F_{L^q}^*(G) = C^*(G)$ . Let  $(\pi, H)$  be a unitary representation of  $G$  and let  $(\pi_{\mathbb{R}}, H_{\mathbb{R}})$  be the orthogonal representation obtained by restriction of scalars. It is clear that  $\|\pi(f)\| = \|\pi_{\mathbb{R}}(f)\|$ , for any  $f \in L^1(G)$ . By Corollary A.7.15 in [BLV08], we may find a probability space  $(\Omega, \mu)$  and a p.m.p. action  $G \overset{\sigma}{\curvearrowright} (\Omega, \mu)$  such that  $\pi_{\mathbb{R}}$  is contained in  $\pi_{2,\sigma|_{\mathbb{R}}}$ . Here, the latter is the representation  $\pi_{2,\sigma}$  on  $L^2(\Omega, \mu)$  associated with the action  $\sigma$  and the trivial 1-cocycle, but viewed as an orthogonal representation on  $L^2(\Omega, \mu; \mathbb{R})$  via restriction of scalars. Thus, for each  $f \in L^1(G)$ , we have  $\|\pi(f)\| \leq \|\pi_{2,\sigma}(f)\|$ . Together with inequality (IV.10), this implies that  $\|\pi(f)\| \leq \|f\|_{F_{L^p}^*}$ . Since  $\pi$  was an arbitrary unitary representation, it follows that  $\|f\|_u \leq \|f\|_{F_{L^p}^*}$ , where  $\|\cdot\|_u$  denotes the norm on  $C^*(G)$ . As  $f \in L^1(G)$  was arbitrary, the claim follows also in this case.  $\square$

*Remark IV.3.4.* The assumption that  $G$  is second countable in Theorem IV.3.3 is needed to employ Corollary A.7.15 from [BLV08] in the case  $q = 2$ . When  $1 \leq p < q < 2$ , the assumption of second countability can be removed.

## IV.4 The dual of a symmetrized pseudofunction algebra

Let  $(\pi, E)$  be an isometric representation of the locally compact group  $G$  on a reflexive Banach space  $E$ . The goal of this section is to describe the dual of the symmetrized  $\pi$ -pseudofunction algebra,  $F_\pi^*(G)$ . We shall prove Theorem C from the introduction, restated here as Theorem IV.4.3, which identifies  $F_\pi^*(G)'$  with the sumspace of the  $\pi$ - and  $\pi'$ -Fourier-Stieltjes spaces. This is parallel to the non-symmetrized setting where  $F_\pi(G)'$  can be identified with the  $\pi$ -Fourier-Stieltjes space – a fact which we recall in Proposition IV.4.1.

Denote by  $\langle \cdot, \cdot \rangle$  the duality pairing between  $L^1(G)$  and  $L^\infty(G)$  given, for  $f \in L^1(G)$  and  $\varphi \in L^\infty(G)$ , by

$$\langle f, \varphi \rangle = \int_G f(s)\varphi(s) \, d\mu_G(s),$$

The map  $L^\infty(G) \rightarrow L^1(G)'$  given by  $\varphi \mapsto \langle \square, \varphi \rangle$  is a linear isometric isomorphism. Given an isometric representation  $(\pi, E)$  of  $G$ , the canonical linear contraction from  $L^1(G)$  to  $F_\pi(G)$  has dense range by construction. Hence,  $F_\pi(G)'$  embeds contractively into  $L^1(G)'$ . We use the identification of  $L^1(G)'$  and  $L^\infty(G)$  in Proposition IV.4.1 to identify  $F_\pi(G)'$  with the  $\pi$ -Fourier-Stieltjes space, which we recall embeds contractively into  $L^\infty(G)$ .

**Proposition IV.4.1.** *Let  $(\pi, E)$  be an isometric Banach space representation of the locally compact group  $G$ . The canonical identification of  $L^1(G)'$  with  $L^\infty(G)$  restricts to an isometric isomorphism between  $F_\pi(G)'$  and  $B_\pi(G)$ .*

*Proof.* Let  $\varphi \in B_\pi(G)$ . By construction of  $B_\pi(G)$ , we have  $|\langle f, \varphi \rangle| \leq \|\varphi\|_{B_\pi} \|\pi(f)\|$ , for all  $f \in L^1(G)$ . Therefore,  $\langle \square, \varphi \rangle$  extends to a functional  $\zeta_\varphi$  on  $F_\pi(G)$  with  $\|\zeta_\varphi\| \leq \|\varphi\|_{B_\pi}$ . Conversely, for each  $\zeta \in F_\pi(G)'$  there exists a unique  $\varphi_\zeta \in L^\infty(G)$  such that  $\zeta(f) = \langle f, \varphi_\zeta \rangle$ , for all  $f \in L^1(G)$ . Then  $|\langle f, \varphi_\zeta \rangle| \leq \|\zeta\| \|\pi(f)\|$ , for all  $f \in L^1(G)$ . Hence,  $\varphi_\zeta$  lies in  $B_\pi(G)$  and  $\|\varphi_\zeta\|_{B_\pi} \leq \|\zeta\|$ .  $\square$

*Remark IV.4.2.* Together with Theorem IV.2.4, Proposition IV.4.1 implies that  $F_\pi(G)'$  embeds contractively into the  $p$ -Fourier-Stieltjes algebra  $B_p(G)$ , and further, that this canonical embedding is an isometric isomorphism when  $\pi$  is  $p$ -universal. This is the content of Theorem 6.6 in [Run04].

Let in the following  $(\pi, E)$  be an isometric representation of the locally compact group  $G$  on a reflexive Banach space  $E$ . We turn our attention to the symmetrized  $\pi$ -pseudofunction algebra,  $F_\pi^*(G)$ .

**Theorem IV.4.3.** *Let  $(\pi, E)$  be an isometric representation of the locally compact group  $G$  on a reflexive Banach space  $E$ . The canonical identification of  $L^1(G)'$  with  $L^\infty(G)$  restricts to an isometric isomorphism between  $F_\pi^*(G)'$  and the sum space  $B_\pi(G) + B_{\pi'}(G)$ .*

*Proof.* The direct sum of  $\pi$  and its dual representation  $\pi'$  defines in a natural way a linear map  $\pi \oplus \pi' : L^1(G) \rightarrow \mathcal{L}(E) \oplus \mathcal{L}(E')$  given by  $(\pi \oplus \pi')(f) = \pi(f) \oplus \pi'(f)$ . By construction of  $F_\pi^*(G)$ , this extends to a linear isometry  $\pi \oplus \pi' : F_\pi^*(G) \rightarrow \mathcal{L}(E) \oplus_\infty \mathcal{L}(E')$  whose image is a closed subspace. We identify the dual of  $\mathcal{L}(E) \oplus_\infty \mathcal{L}(E')$  with  $\mathcal{L}(E)' \oplus_1 \mathcal{L}(E)'$  and denote by  $(\pi \oplus \pi')^* : \mathcal{L}(E)' \oplus_1 \mathcal{L}(E)'$   $\rightarrow F_\pi^*(G)'$  the Banach space adjoint of  $\pi \oplus \pi'$ . Concretely,  $(\pi \oplus \pi')^*$  is given by, for  $\Phi \in \mathcal{L}(E)'$ ,  $\Psi \in \mathcal{L}(E)'$  and  $x \in F_\pi^*(G)$ ,

$$(\pi \oplus \pi')^*(\Phi \oplus \Psi)(x) = \Phi \oplus \Psi((\pi \oplus \pi')(x)) = \Phi(\pi(x)) + \Psi(\pi'(x)).$$

Because  $\pi \oplus \pi'$  is an isometric isomorphism onto its image,  $(\pi \oplus \pi')^*$  considered as a map  $(\text{Im}(\pi \oplus \pi'))' \rightarrow F_\pi^*(G)'$  is an isometric isomorphism by Lemma 3.6 in [GT14]. Let  $\zeta \in F_\pi^*(G)'$  and let  $\varphi_\zeta$  be the unique function in  $L^\infty(G)$  such that  $\zeta(f) = \langle f, \varphi_\zeta \rangle$ , for  $f \in L^1(G)$ . By the Hahn-Banach Extension Theorem, we can find  $\Phi_\zeta \in \mathcal{L}(E)'$  and  $\Psi_\zeta \in \mathcal{L}(E)'$  such that

$$(\pi \oplus \pi')^*(\Phi_\zeta \oplus \Psi_\zeta) = \zeta \quad \text{and} \quad \|\zeta\| = \|\Phi_\zeta\| + \|\Psi_\zeta\|.$$

Then  $\Phi_\zeta \circ \pi + \Psi_\zeta \circ \pi'$ , viewn as a function on  $G$ , satisfies, for all  $f \in L^1(G)$ ,

$$\zeta(f) = \langle f, \Phi_\zeta \circ \pi + \Psi_\zeta \circ \pi' \rangle.$$

Thus,  $\Phi_\zeta \circ \pi + \Psi_\zeta \circ \pi'$  equals  $\varphi_\zeta$  by the uniqueness of  $\varphi_\zeta$ . Now  $\Phi_\zeta \circ \pi$  lies in  $B_\pi(G)$  with  $\|\Phi_\zeta \circ \pi\|_{B_\pi} \leq \|\Phi_\zeta\|$ , and  $\Psi_\zeta \circ \pi'$  lies in  $B_{\pi'}(G)$  with  $\|\Psi_\zeta \circ \pi'\|_{B_{\pi'}} \leq \|\Psi_\zeta\|$ . Therefore,  $\varphi_\zeta$  lies in the sum space  $B_\pi(G) + B_{\pi'}(G)$  with

$$\|\varphi_\zeta\|_{B_\pi + B_{\pi'}} \leq \|\Phi_\zeta\| + \|\Psi_\zeta\| = \|\zeta\|.$$

Hence, the canonical map  $L^1(G)' \rightarrow L^\infty(G)$  given by  $\zeta \mapsto \varphi_\zeta$  restricts to a contraction  $F_\pi^*(G)' \rightarrow B_\pi(G) + B_{\pi'}(G)$ .

Conversely, let  $\varphi \in B_\pi(G) + B_{\pi'}(G)$ . For each  $\varepsilon > 0$ , we can find  $\varphi_{1,\varepsilon} \in B_\pi(G)$  and  $\varphi_{2,\varepsilon} \in B_{\pi'}(G)$  so that  $\varphi = \varphi_{1,\varepsilon} + \varphi_{2,\varepsilon}$  and

$$\|\varphi_{1,\varepsilon}\|_{B_\pi} + \|\varphi_{2,\varepsilon}\|_{B_{\pi'}} \leq \|\varphi\|_{B_\pi + B_{\pi'}} + \varepsilon.$$

Then, for  $f \in L^1(G)$ ,

$$\begin{aligned} |\langle f, \varphi \rangle| &\leq |\langle f, \varphi_{1,\varepsilon} \rangle| + |\langle f, \varphi_{2,\varepsilon} \rangle| \\ &\leq \|\varphi_{1,\varepsilon}\|_{B_\pi} \|\pi(f)\| + \|\varphi_{2,\varepsilon}\|_{B_{\pi'}} \|\pi'(f)\| \\ &\leq \left( \|\varphi\|_{B_\pi + B_{\pi'}} + \varepsilon \right) \|f\|_{F_\pi^*}. \end{aligned}$$

Since  $\varepsilon > 0$  was arbitrary, it follows that  $\langle \square, \varphi \rangle$  extends to a functional on  $F_\pi^*(G)$  with  $\|\langle \square, \varphi \rangle\| \leq \|\varphi\|_{B_\pi + B_{\pi'}}$ . Hence, the canonical map  $L^\infty(G) \rightarrow L^1(G)'$  given by  $\varphi \mapsto \langle \square, \varphi \rangle$  restricts to a contraction  $B_\pi(G) + B_{\pi'}(G) \rightarrow F_\pi^*(G)'$ . Because  $\varphi \mapsto \langle \square, \varphi \rangle$  and  $\zeta \mapsto \varphi_\zeta$  are inverse of each other, this finishes the proof.  $\square$

Since  $B_\pi(G)$  embeds into  $B_p(G)$ , for any  $QSL^p$ -representation  $\pi$ , we obtain Corollary IV.4.4 as an immediate corollary to Theorem IV.4.3.

**Corollary IV.4.4.** *Let  $(\pi, E)$  be an isometric  $QSL^p$ -representation of the locally compact group  $G$ , for  $1 < p < \infty$ . Then  $F_\pi^*(G)'$  embeds contractively into the sumspace  $B_p(G) + B_{p'}(G)$ .*

## IV.5 Applications to amenability

In this section, we apply the understanding obtained in Section IV.4 of the Banach space dual of a symmetrized pseudofunction algebra in order to prove Theorem B from the introduction. This theorem, which we restate as Theorem IV.5.1 below, is parallel to, and builds upon, analogous characterizations of amenability in terms of the  $p$ -pseudofunction algebra of Herz. We shall give an overview of these classical results as we need them.

**Theorem IV.5.1.** *Let  $G$  be a locally compact group and let  $1 < p, p' < \infty$  be Hölder conjugates. The following are equivalent:*

- (i)  $G$  is amenable,
- (ii)  $F_{\lambda_p}^*(G)'$  is canonically isometrically isomorphic to  $B_p(G) + B_{p'}(G)$ ,
- (iii) The canonical map  $F_{L^p}^*(G) \rightarrow F_{\lambda_p}^*(G)$  is an isometric isomorphism,
- (iv) The canonical map  $F_{L^p}^*(G) \rightarrow F_{\lambda_p}^*(G)$  is an isomorphism,
- (v) The trivial representation  $1_G$  extends to a  $*$ -representation of  $F_{\lambda_p}^*(G)$ .

The conditions (ii) and (v) in Theorem IV.5.1 describes properties of the Banach space dual of  $F_{\lambda_p}^*(G)$ . In the non-symmetrized setting, a connection between amenability of the group  $G$  and similar properties of the Banach space dual of the  $p$ -pseudofunction algebra was first established by Cowling in Theorem 5 in [Cow79] and later refined by Runde in Theorem 6.7 in [Run04] and Neufang and Runde in Theorem 4.1 in [NR07].

**Theorem IV.5.2** (Cowling [Cow79], Runde [Run04], Neufang & Runde [NR07]). *Let  $G$  be a locally compact group. The following are equivalent:*

- (i)  $G$  is amenable,
- (ii)  $F_{\lambda_p}(G)'$  is canonically isometrically isomorphic to  $B_p(G)$ ,
- (iii) The trivial representation  $1_G$  extends to a representation of  $F_{\lambda_p}(G)$ .

We build upon this work in the non-symmetrized setting and apply Theorem IV.4.3 to prove the first implication in Theorem IV.5.1.

*Proof of Theorem IV.5.1 (i)⇒(ii).* Assume  $G$  is amenable. Then, by Theorem IV.5.2,  $F_{\lambda_q}(G)'$  is canonically isometrically isomorphic to  $B_q(G)$ , for each  $1 < q < \infty$ . Together with Theorem IV.4.3 with  $\lambda_p$  in place of  $\pi$ , this implies that

$$F_{\lambda_p}^*(G)' \cong F_{\lambda_p}(G)' + F_{\lambda_{p'}}(G)' \cong B_p(G) + B_{p'}(G),$$

where both isomorphisms are canonical and isometric.  $\square$

In  $C^*$ -algebraic terms, amenability of  $G$  is the property that its universal and reduced group  $C^*$ -algebras coincide canonically. In Theorem 3.7 in [GT14], Gardella and Thiel give an  $L^p$ -generalization of this. Here, the roles of the universal and reduced  $C^*$ -algebras are played by the  $L^p$ -, respectively,  $p$ -pseudofunction algebras.

**Theorem IV.5.3** (Gardella & Thiel [GT14]). *Let  $G$  be a locally compact group and let  $1 < p < \infty$ . The following are equivalent:*

- (i)  $G$  is amenable,
- (ii) The canonical map  $F_{L^p}(G) \rightarrow F_{\lambda_p}(G)$  is an isometric isomorphism,
- (iii) The canonical map  $F_{L^p}(G) \rightarrow F_{\lambda_p}(G)$  is an isomorphism.

In our Theorem IV.5.1, the equivalence of amenability and the conditions (iii) and (iv) is parallel to this result of Gardella and Thiel, and our proof of the implications (ii)⇒(iii) and (iv)⇒(v) is an adaptation of theirs. The implication (iii)⇒(iv) in Theorem IV.5.1 is trivial.

*Proof of Theorem IV.5.1 (ii)⇒(iii).* By Theorem IV.4.3, the sumspace  $B_p(G) + B_{p'}(G)$  can be identified canonically with the dual of  $F_{QSL^p}^*(G)$ . Now the canonical inclusion of  $F_{\lambda_p}^*(G)'$  into the sum space  $B_p(G) + B_{p'}(G)$  is the Banach space adjoint of the canonical linear contraction from  $F_{QSL^p}^*(G)$  to  $F_{\lambda_p}^*(G)$ . Thus, it follows from Lemma 3.6 in [GT14] that if  $F_{\lambda_p}^*(G)' \hookrightarrow B_p(G) + B_{p'}(G)$  is an isometric isomorphism then so is  $F_{QSL^p}^*(G) \rightarrow F_{\lambda_p}^*(G)$ . Since  $F_{L^p}^*(G)$  is intermediate to  $F_{QSL^p}^*(G)$  and  $F_{\lambda_p}^*(G)$ , we see that (iii) follows from (ii).  $\square$

*Proof of Theorem IV.5.1 (iv)⇒(v).* The trivial representation  $1_G$  is an  $L^p$ -representation. Thus, it extends to a  $*$ -representation of  $F_{L^p}^*(G)$ . Therefore, if the canonical map  $F_{L^p}^*(G) \rightarrow F_{\lambda_p}^*(G)$  is an isomorphism,  $1_G$  extends to a  $*$ -representation of  $F_{\lambda_p}^*(G)$ .  $\square$

Due to work of Hulanicki in [Hul66] and Reiter in [Rei65], amenability of  $G$  is characterized as the property that the trivial representation  $1_G$  is weakly contained in the left regular representation  $\lambda_2$  of  $G$  on  $L^2(G)$ , or, equivalently, that  $1_G$  extends to a  $*$ -representation of  $C_r^*(G)$ . The characterization of amenability in Theorem IV.5.2(iii) should be understood as an  $L^p$ -version of the characterization by Hulanicki and Reiter. Indeed, for  $p = 2$ , the 2-pseudofunction algebra  $F_{\lambda_2}(G)$  is nothing but the reduced group  $C^*$ -algebra of  $G$ . In the same way, Theorem IV.5.1(v) is another  $L^p$ -version of this characterization but this time in the symmetrized setting. The proof of (v)⇒(i) in Theorem IV.5.1 is due to work

of Samei & Wiersma in Proposition 3.1 in [SW24]. We shall include their proof for the sake of completeness and in order to add to it a few points of clarification. The proof relies on a theorem due to Kesten in [Kes59] (see also Theorem G.4.4 in [BLV08]).

**Theorem IV.5.4** (Kesten [Kes59]). *Let  $G$  be a locally compact group and let  $\pi$  be a unitary representation of  $G$ .*

- (i) *If  $1_G \prec \pi$  then  $\|\pi(f)\| = 1$ , for all  $f \in L^1(G)$  with  $f \geq 0$  and  $\|f\|_1 = 1$ .*
- (ii) *If there exists  $f \in L^1(G)$  with  $f \geq 0$ ,  $\|f\|_1 = 1$  and  $\|\pi(f)\| = 1$ , and such that  $\text{supp}(f^* * f)$  generates a dense subgroup of  $G$ , then  $1_G \prec \pi$ .*

When  $G$  is  $\sigma$ -finite, we get the following characterization of amenability as a corollary to Kesten's theorem. We assume this is well known to experts.

**Corollary IV.5.5.** *Let  $G$  be a locally compact group. Then  $G$  is amenable if and only if  $\|\lambda_2(f)\| = 1$ , for all  $f \in L^1(G)$  with  $f \geq 0$  and  $\|f\|_1 = 1$ .*

*Proof.* Suppose  $G$  is amenable so that  $1_G$  is weakly contained in  $\lambda_2$ . Then, it follows directly from Theorem IV.5.4(i) that  $\|\lambda_2(f)\| = 1$ , for all  $f \in L^1(G)$  with  $f \geq 0$  and  $\|f\|_1 = 1$ . For the converse implication, we consider first the case where  $G$  is  $\sigma$ -finite. Assume that  $G$  is not amenable so that  $1_G$  is not weakly contained in  $\lambda_2$ . As  $G$  is  $\sigma$ -finite, it decomposes into a countable union of disjoint sets of finite measure. Write  $G = \bigcup_{n \in \mathbb{N}} F_n$  and define

$$f = \sum_{n=1}^{\infty} \frac{1}{\mu_G(F_n)2^n} 1_{F_n},$$

where, for each  $n \in \mathbb{N}$ ,  $1_{F_n}$  is the indicator function on the set  $F_n$ . Then  $f$  is a non-negative function with  $\|f\|_1 = 1$  and with full support, and so,  $\text{supp}(f^* * f)$  is all of  $G$ , as well. Then, by Theorem IV.5.4(ii), we must have  $\|\lambda_2(f)\| < 1$ .

Finally, let  $G$  be a general (not necessarily  $\sigma$ -finite) locally compact group. Assume that  $\|\lambda_2(f)\| = 1$ , for all  $f \in L^1(G)$  with  $f \geq 0$  and  $\|f\|_1 = 1$ . Then, for every open  $\sigma$ -finite subgroup  $H \leq G$  and every  $g \in L^1(H)$  with  $g \geq 0$  and  $\|g\|_1 = 1$ , we still have  $\|\lambda_2(g)\| = 1$ . Hence, by the above argument, every open  $\sigma$ -finite subgroup of  $G$  is amenable. As  $G$  is the union of its open  $\sigma$ -finite subgroups, it follows that  $G$  is amenable.  $\square$

Let  $f \in L^1(G)$ . For parameters  $1 \leq p_1 < p_2 < p_3 \leq 2$ , a standard interpolation argument based on the Riesz-Thorin theorem yields a bound on the norm of  $f$  viewed as an element of the symmetrized  $p_2$ -pseudofunction algebra in terms of the norms of  $f$  in the symmetrized  $p_1$ - and  $p_3$ -pseudofunction algebras. This was observed and proved by Samei and Wiersma in Proposition 4.5 in [SW20]. We record this fact without proof in Lemma IV.5.6 below.

**Lemma IV.5.6.** *Let  $G$  be a locally compact group. Let  $1 \leq p_1 < p_2 < p_3 \leq 2$  and let  $0 < \theta < 1$  be such that*

$$\frac{1}{p_2} = \frac{1 - \theta}{p_1} + \frac{\theta}{p_3}.$$

Then, for every  $f \in L^1(G)$ ,

$$\|f\|_{F_{\lambda_{p_2}}^*} \leq \|f\|_{F_{\lambda_{p_1}}^*}^{1-\theta} \|f\|_{F_{\lambda_{p_3}}^*}^\theta.$$

We can now give the proof of Samei and Wiersma of the last remaining implication of Theorem IV.5.1. Observe that the symmetrized  $p$ -pseudofunction algebra recovers  $L^1(G)$  when  $p = 1$  and  $C_r^*(G)$  when  $p = 2$ . Hence, for any parameter  $1 < p < 2$ , we can employ Lemma IV.5.6 to get an upper bound of the symmetrized  $p$ -pseudofunction algebra norm in terms of the  $L^1$ -norm and the reduced  $C^*$ -norm.

*Proof of Theorem IV.5.1 (v)  $\Rightarrow$  (i).* Assume that  $G$  is *not* amenable. By Corollary IV.5.5, we may then find a function  $f \in L^1(G)$  such that  $f \geq 0$ ,  $\|f\|_1 = 1$  and  $\|\lambda_2(f)\| < 1$ . As  $f \geq 0$ , we have  $1_G(f) = \|f\|_1 = 1$ . Further, let  $0 < \theta < 1$  be such that  $1/p = (1 - \theta)/1 + \theta/2$ . Lemma IV.5.6 with  $p_1 = 1$ ,  $p_2 = p$  and  $p_3 = 2$  yields the upper bound

$$\|f\|_{F_{\lambda_p}^*} \leq \|f\|_1^{1-\theta} \|\lambda_2(f)\|^\theta < 1.$$

As the involution on  $F_{\lambda_p}^*(G)$  is an isometry, it follows from Theorems 11.1.4-5 in [Pal01] that any  $*$ -representation of  $F_{\lambda_p}^*(G)$  must be a contraction. Hence,  $1_G$  does not extend to a  $*$ -representation of  $F_{\lambda_p}^*(G)$ .  $\square$

**Acknowledgements** The author thanks Nadia Larsen for many discussions, Mikael de la Salle for sharing his insights into complex interpolation theory and Matthew Wiersma for sharing how to extend the proof of Corollary IV.5.5 to the non- $\sigma$ -finite case. The author is grateful to Eusebio Gardella and Hannes Thiel for many useful comments on an earlier draft of this paper that helped improve its presentation.

## Erratum

1. Let  $\xi, \eta \in S(\Omega)$ . For  $z = \theta + i\gamma \in \mathbb{S}$  and  $\omega \in \Omega$ , we can rewrite  $T_z\xi(\omega)$  from (IV.3) as follows:

$$\begin{aligned} T_z\xi(\omega) &= \int_G f(s)c_s(\omega) \left( \frac{ds \cdot \mu}{d\mu}(\omega) \right)^{\gamma(1/p_0 - 1/p_1)i + \theta/p_0 + (1-\theta)/p_1} \xi(s^{-1} \cdot \omega) d\mu_G(s) \\ &= \int_G f(s)c_s(\omega) \left( \frac{ds \cdot \mu}{d\mu}(\omega) \right)^{z(1/p_0 - 1/p_1)} \left( \frac{ds \cdot \mu}{d\mu}(\omega) \right)^{1/p_1} \xi(s^{-1} \cdot \omega) d\mu_G(s). \end{aligned}$$

Analyticity on  $\mathbb{S}^\circ$  of the map  $z \mapsto \int_\Omega (T_z\xi)\eta d\mu$  defined in (IV.4) is easily seen from this.

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